

Formation ATIAM

Acoustique Traitement du Signal Informatique Appliqués à la Musique Parcours
multi-mentions du Master (M2) Sciences et Technologies de l'Université Pierre et
Marie Curie (Paris 6)

UE Méthodes mathématiques pour l'informatique musicale [MMIM] - 2011-2012

Moreno Andreatta
Equipe Représentations Musicales
IRCAM/CNRS/UPMC UMR 9912



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Marie Curie (Paris 6)

MODELES MATHEMATIQUES POUR L'INFORMATIQUE MUSICALE (MMIM) : OUTILS THEORIQUES ET STRATEGIES PEDAGOGIQUES

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Équipe Représentations Musicales

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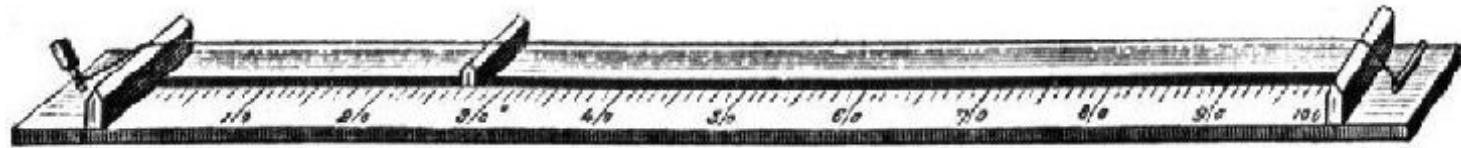
Andreatta M. et M. Chemillier (2007), « Modèles mathématiques pour l'informatique musicale (MMIM): Outils théoriques et stratégies pédagogiques », Actes des Journées d'Informatique Musicale, Lyon, avril, p. 113-12
➔ <http://articles.ircam.fr/textes/Andreatta07b/index.pdf>

Mathématiques/Musique...une histoire récente!

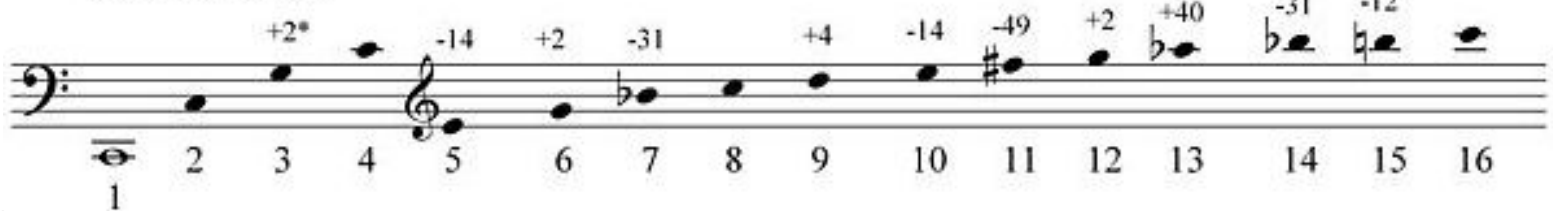
- 1999 : 4^e Forum Diderot (Paris, Vienne, Lisbonne), *Mathematics and Music* (G. Assayag, H.G. Feichtinger, J.F. Rodrigues, Springer, 2001)
- 2000-2001 : Séminaire *MaMuPhi*, *Penser la musique avec les mathématiques ?* (Assayag, Mazzola, Nicolas éd., Coll. « M/S », Ircam/Delatour, 2006)
- 2000-2003 : International Seminar on *MaMuTh* (*Perspectives in Mathematical and Computational Music Theory*) (Mazzola, Noll, Luis-Puebla eds, epOs, 2004)
- 2003 : *The Topos of Music* (G. Mazzola et al.)
- 2003: *Music and Mathematics. From Pythagoras to Fractals* (J. Fauvel et al.)
- 2001 - 2011 : Séminaire *MaMuX* de l'Ircam
- 2004 - 2011 : Séminaire *mamuphi* (Ens/Ircam)
- 2006 : *Mathematical Theory of Music* (F. Jędrzejewski), Coll. « M/S »
- 2007 : *La vérité du beau dans la musique* (G. Mazzola), Coll. « Musique/Sciences »
- 2007 : *Journal of Mathematics and Music* (Taylor & Francis) et MCM 2007
- 2007: *Music. A Mathematical Offering* (Dave Benson), CUP
- 2008: *Music Theory and Mathematics* (Jack Douthett et al.), URP
- 2009 : *Computational Music Science Series* (Springer)
- 2009 : MCM 2009 (Yale) et Proceedings chez Springer
- 2010 : Mathematics Subject Classification : 00A65 Mathematics and music
- 2011 : Conférence de la SMCM (Ircam, 15-17 juin 2011)



De Pythagore... à la théorie des groupes



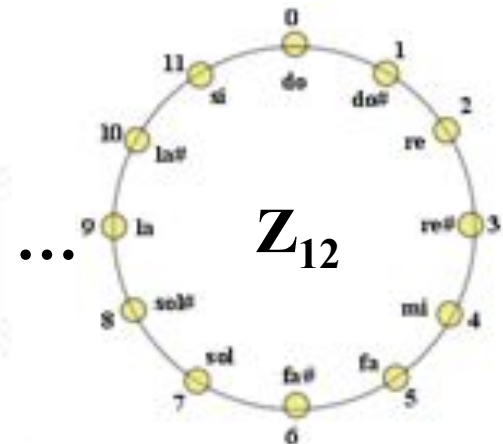
i suoni armonici



Physique

* in cents, confrontati con la scala temperata

Mathématiques

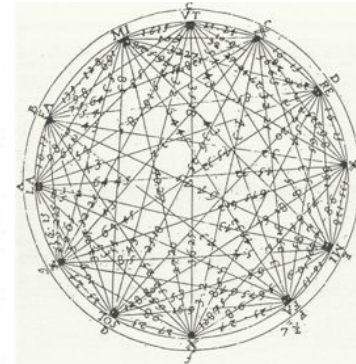


Musique et mathématiques : deux destinées parallèles

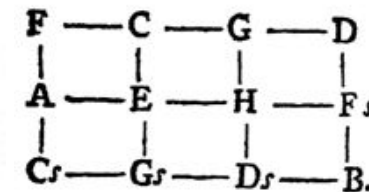
MUSIQUE	MATHS
500 av. J. C. Relation hauteur/longueur corde. La musique est source d'inspiration pour la théorie des nombres et la géométrie.	Nombres naturels et rationnels
300 a.J. Invention (théorique) de la gamme chromatique tempérée égale par Aristoxène de Tarente) et prémonition de la théorie des groupes . Isomorphismes entre les logarithmes (intervalles musicaux) et les exponentiels (longueur d'une corde)	Aucune relation.
1000 Invention de la représentation bidimensionnelle des hauteurs	Aucune correspondance
1500 Aucune reprise des concepts précédents	Nombres négatifs. Construction des rationnels
1600 Aucune relation	Nombres réels et les logarithmes
Martin Mersenne (1588-1648) : combinatoire musicale	Calcul des probabilités
1700 La fugue comme un automate abstrait. Manipulation inconsciente du groupe de Klein	Nombres complexes (Euler, Gauss), les quaternions (Hamilton), continuité (Cauchy), structure de groupe (Galois, Abel)
Leonhard Euler : Speculum Musicum (1773)	Théorie des graphes
1900 Libération de la prison de la tonalité (Loquin, Hauer, Schoenberg)	Nombres infinis et transfinis (Cantor). Axiomatique de Peano. Théorie de la mesure (Lebesgue, Borel)
1920 Formalisation radicale des macrostructures à travers le système sériel (Schoenberg)	Aucun développement de la théorie des nombres.
Ernst Krenek (1900-1991) : les axiomes dans le système dodécaphonique	David Hilbert, Les fondements de la géométrie (1899)



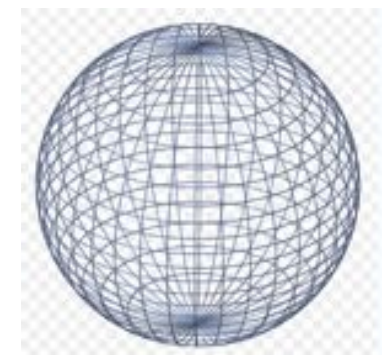
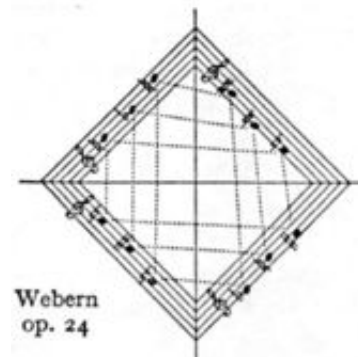
Pythagore et le monochorde, VI^e-V^e siècle av. J. C.



Mersenne, *Harmonicorum Libri XII*, 1648



Euler : *Speculum musicum*, 1773



Iannis Xenakis, *Musique. Architecture*, Tournai, Casterman, 1971, (New, revised edition: Tournai, Casterman, 1976, 238 p.)



Les mathématiques modernes et la création des concepts

MATH / MUSIC MEETINGS

Creativity in Music and Mathematics

Pierre Boulez & Alain Connes

Encounter with two major figures of musical creation and contemporary mathematical research: Pierre Boulez and Alain Connes.

What is the role of intuition in mathematical reasoning and in artistic activities? Is there an aesthetic dimension to mathematical activity? Does the notion of elegance of a mathematical demonstration or of a theoretical construction in music play a role in creativity?



G rard Assayag, director of the CNRS/IRCAM Laboratory for The Science and Technology of Music and Sound, will lead this dialogue on invention in the two disciplines.

Photo: Pierre Boulez   Jean Radel

Wednesday, June 15, 2011, 6:30pm / IRCAM, Espace de projection

Creativity in Mathematics and the Arts

Do mathematics have a unique place within scientific disciplines, as music does within artistic practices?

Starting from the mathematics/music relationship, this final round-table will raise the issue of the connections between art and science.

11 am-1pm Round-Table Discussion

   Palais de la d couverte, Salle de conf rences

With **Jean-Paul Allouche**, mathematician | **Claude Bruter**, mathematician and president of the ESMA | **Yves Hellegouarch**, mathematician | **Tom Johnson**, composer | **Jean-Marc L vy-Leblond**, physicist and author | **Jacques Mandelbrojt**, painter and theoretical physicist | **Jean-Claude Risset**, physicist and composer

Session led by **Moreno Andreatta** (researcher IRCAM/CNRS and vice president of the Society for Mathematics and Computation in Music).



Alain Connes: “...the role of mathematics, which at the beginning was considered as a part of physics, has become – thanks to modern mathematics – a kind of substitution of philosophy with respect to the creation of concepts”.

Double mouvement d'une dynamique mathémusicale

[Cf. M. Andreatta : *Mathematica est exercitium musicae*, HDR, octobre 2010]

MATHEMATIQUES

énoncé
mathématique

généralisation

théorème
général

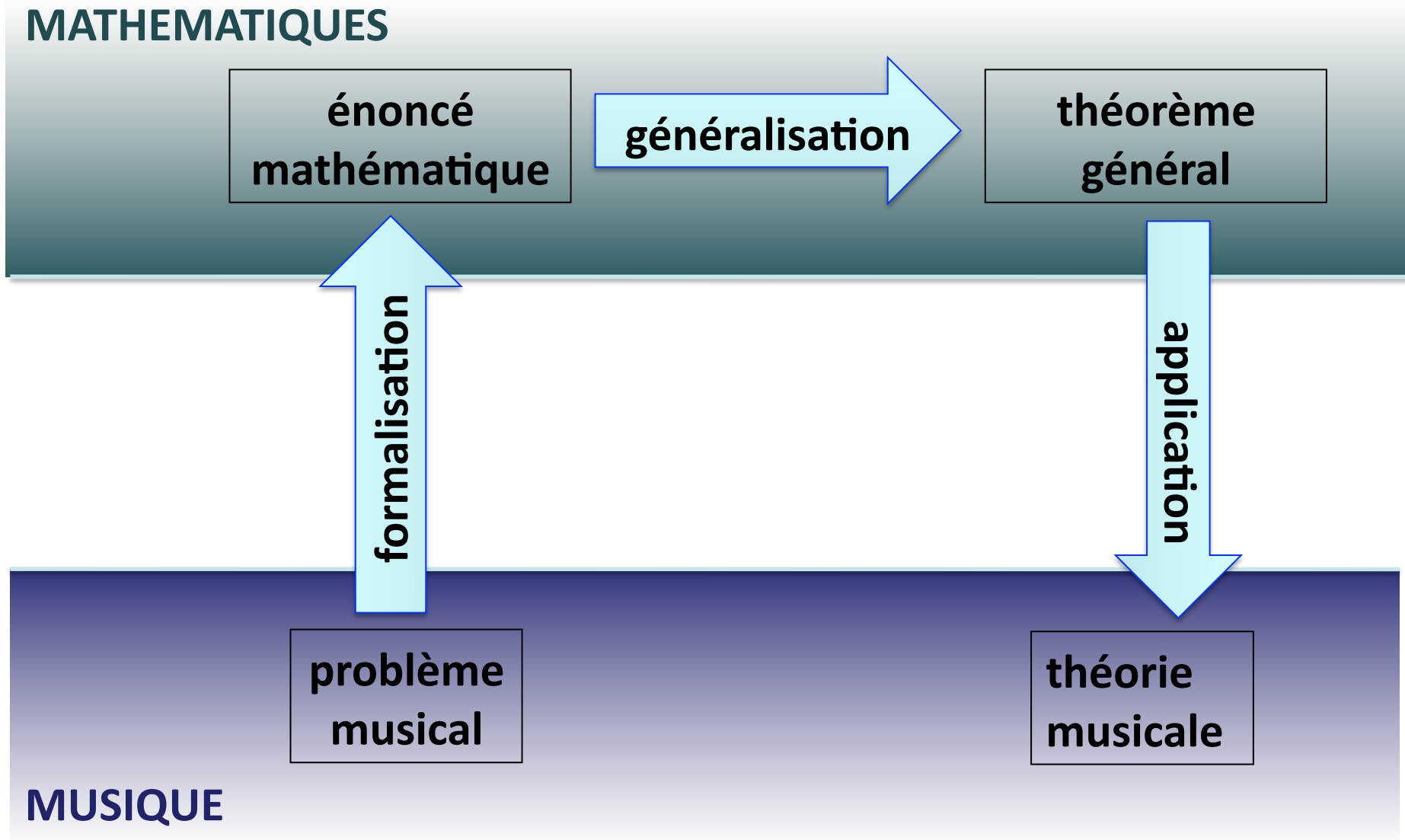
formalisation

application

problème
musical

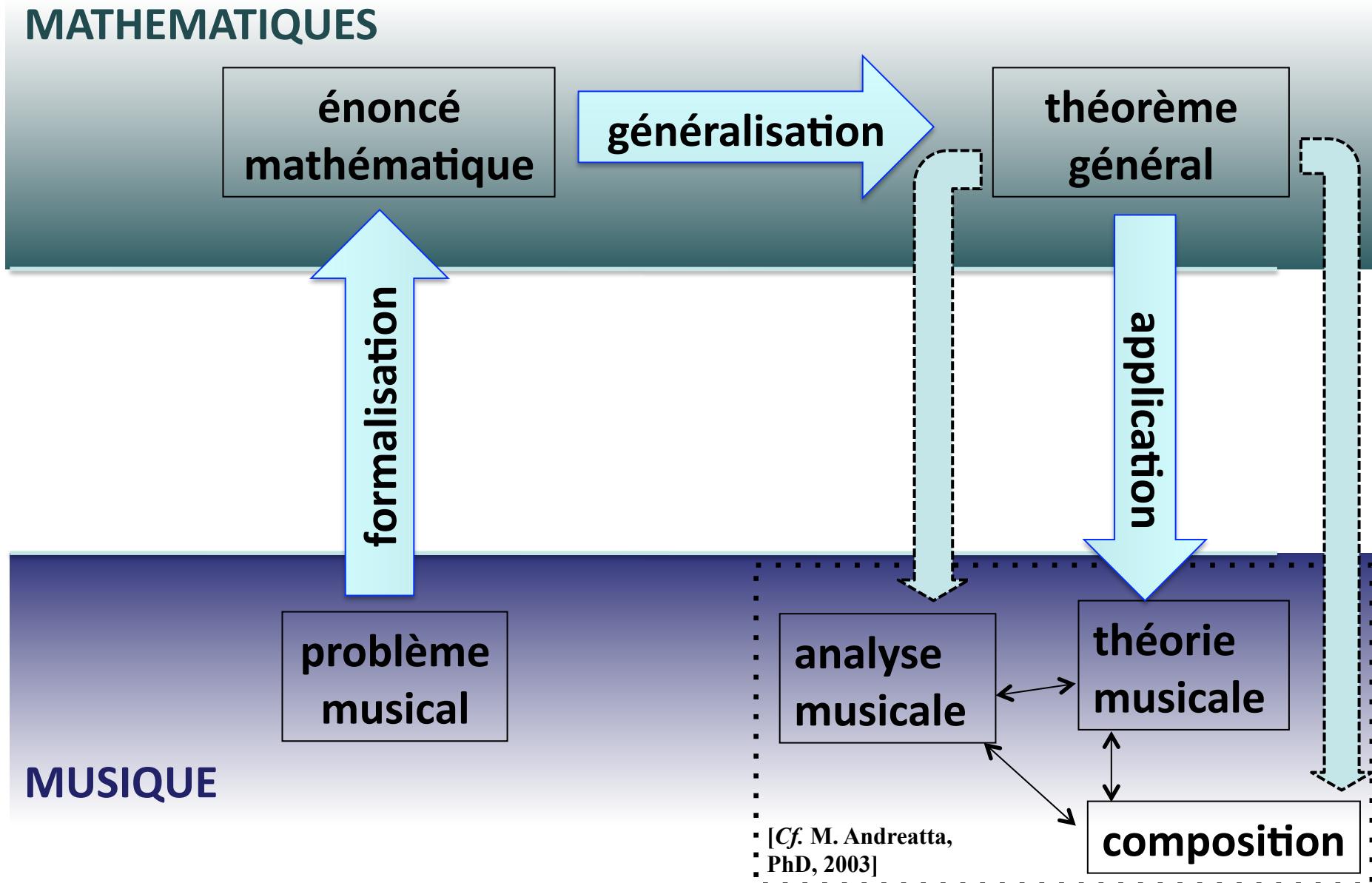
théorie
musicale

MUSIQUE



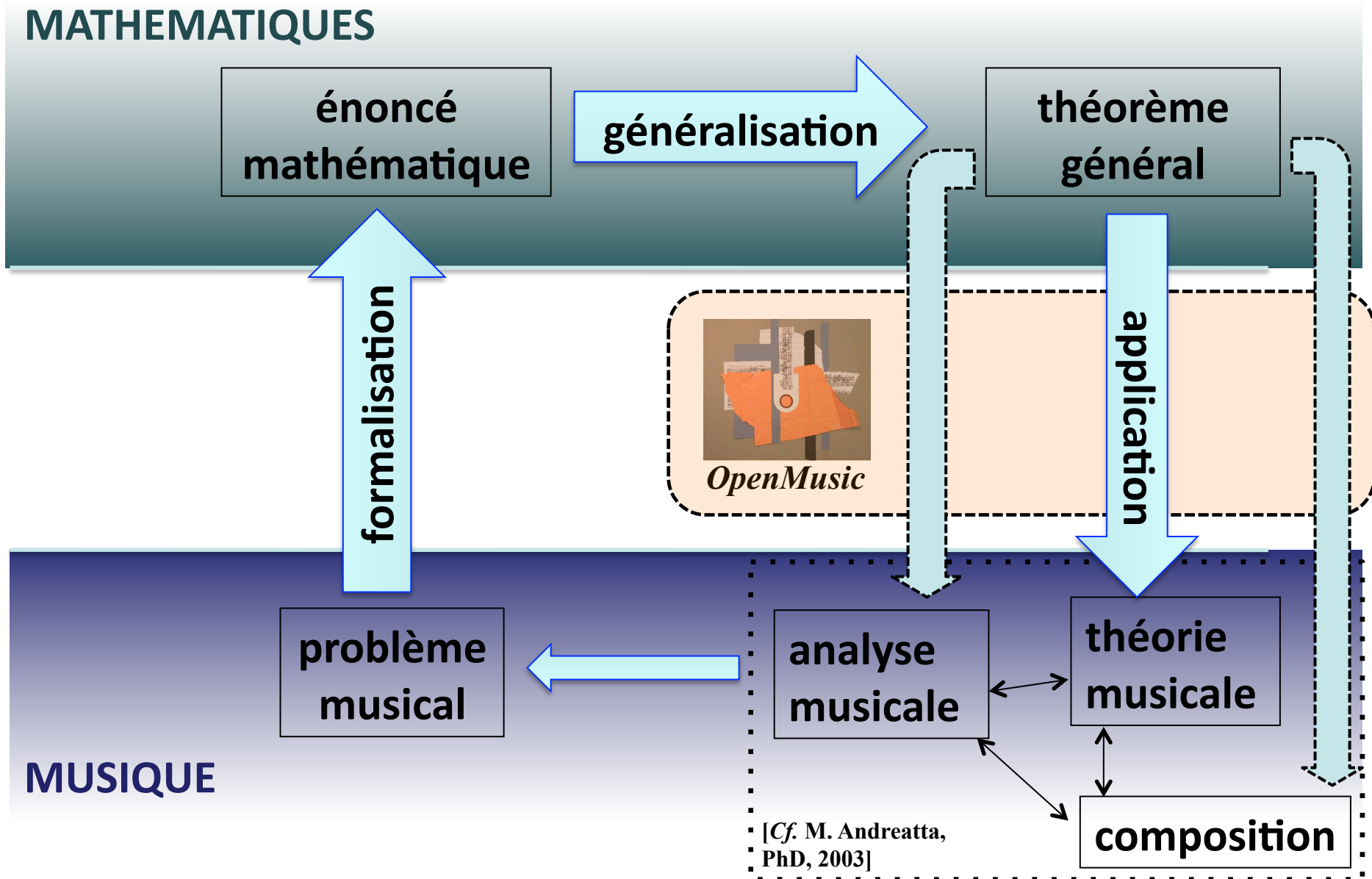
Double mouvement d'une dynamique mathémusicale

[Cf. M. Andreatta : *Mathematica est exercitium musicae*, HDR, octobre 2010]



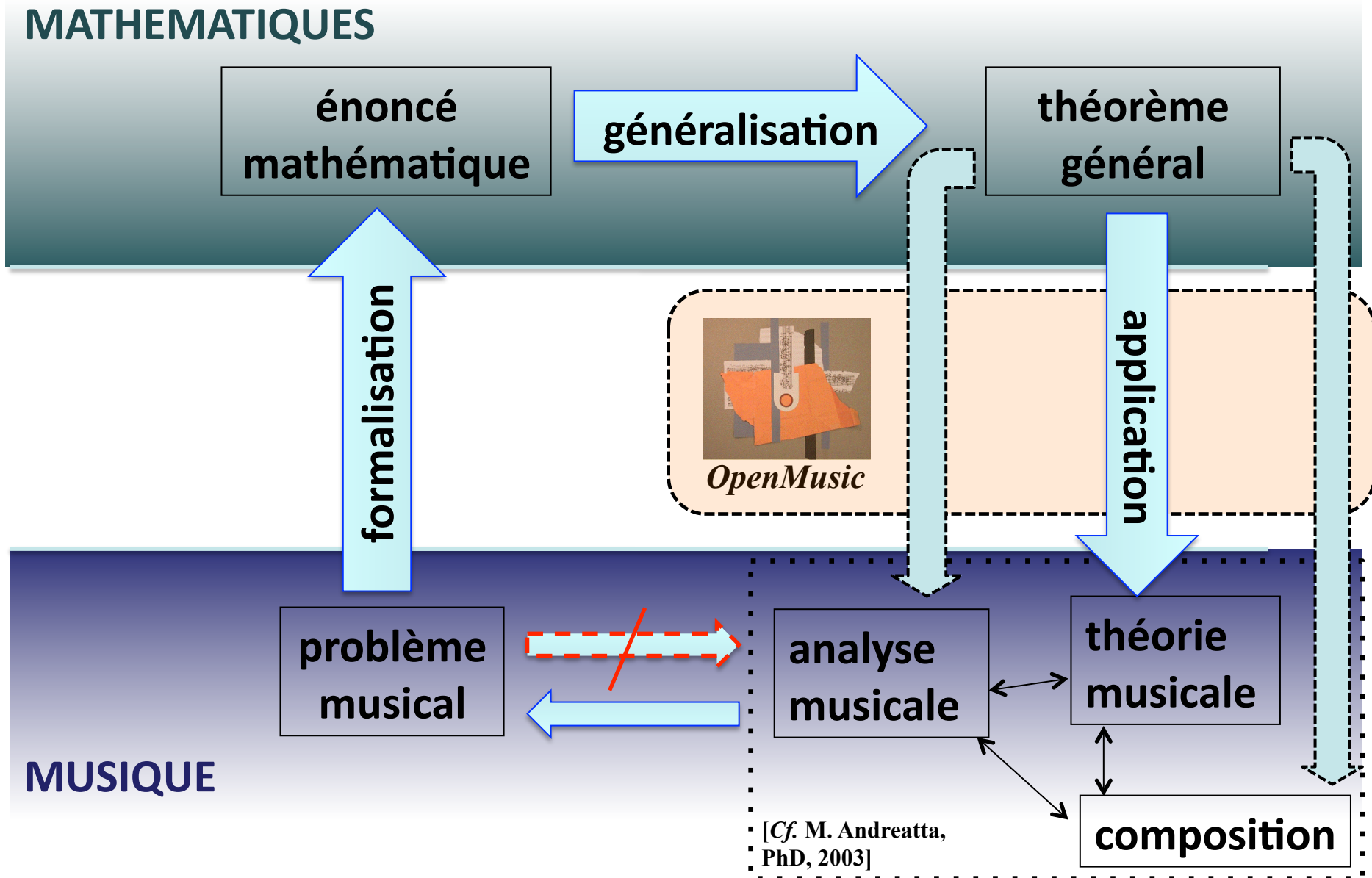
Double mouvement d'une dynamique mathémusicale

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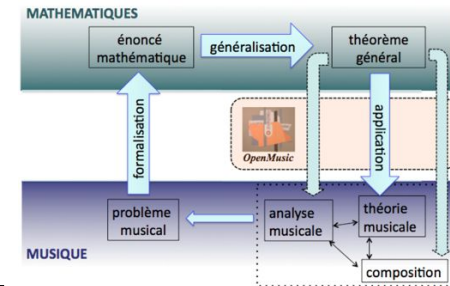
Double mouvement d'une dynamique mathémusicale

[Cf. M. Andreatta : *Mathematica est exercitium musicae*, HDR, octobre 2010]



Plan du cours

1. La construction des canons rythmiques mosaïques : de Minkowski à Fuglede
2. La relation Z et la théorie des ensembles homométriques
3. La *Set Theory* et la théorie transformationnelle
4. Les théories diatoniques et les *ME-sets*
5. Suites périodiques et calcul de différences finies
6. La théorie des block-designs en composition algorithmique
7. Modèles algébriques et catégoriels pour la cognition musicale



Canons rythmiques mosaïques

Relation Z et ensembles homométriques

18 → (0 1 4 6) → [111111] → 4-Z15

23 → (0 1 3 7) → [111111] → 4-Z29

$Df(x)=f(x)-f(x-1).$

7 11 10 11 7 2 7 11 10 11 7 2 7 11...

4 11 1 8 7 5 4 11 1 8 7 5 4 11...

7 2 7 11 10 11 7 2 7 11 10 11...

7 5 4 11 1 8 7 5 4 11 1 8...

.....

Calcul des différences finies

Set Theory, théories transformationnelle et neo-riemanniennes

Théories diatoniques et ME-sets

Block-designs

Quelques notions algébriques utilisées dans le cours

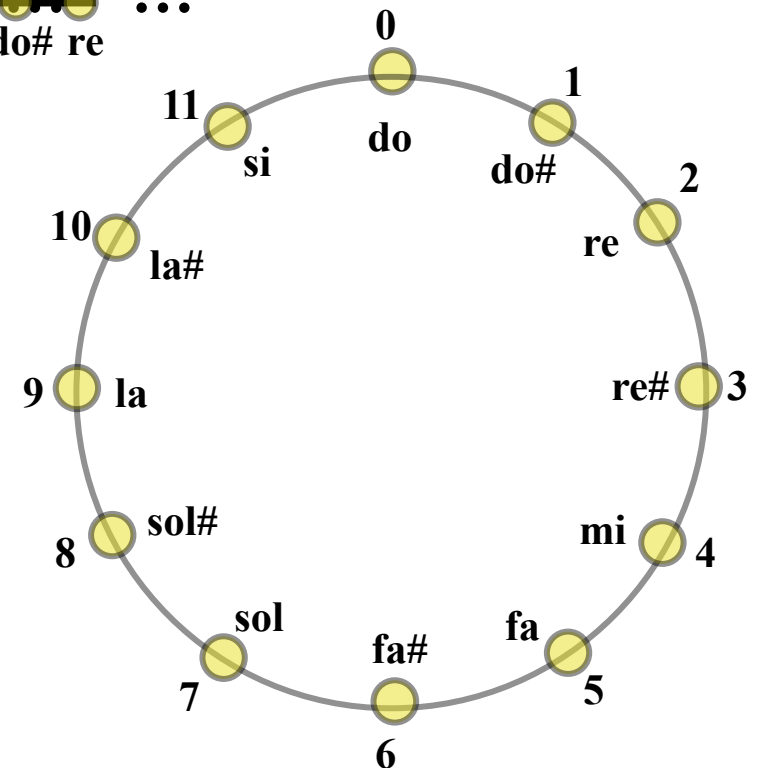
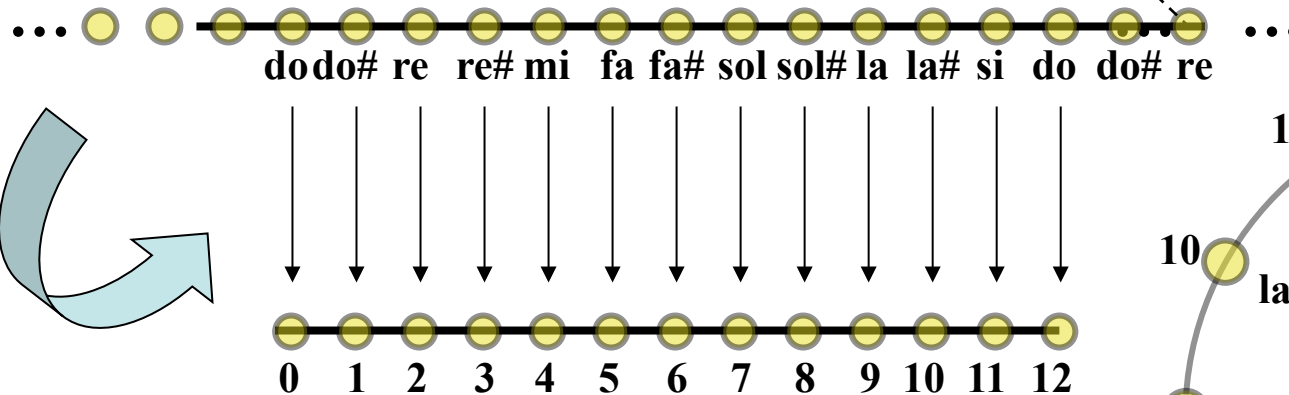
- Définition de groupe
 - Groupe cyclique $\mathbf{Z}/n\mathbf{Z}$ d'ordre n
 - Groupe diédral \mathbf{D}_{2n} d'ordre $2n$
 - Groupe affine \mathbf{Aff}_n d'ordre $\varphi(n) \cdot n$
 - Groupe symétrique \mathbf{S}_n d'ordre $n!$
 - Groupe de Klein
- Action d'un groupe sur un ensemble
 - Relations d'équivalence et orbites
 - Stabilisateur et fixateur d'un élément
 - Lemme de Burnside
 - Action transitive et simplement transitive (GIS)
 - Centralisateur et actions duales
- Factorisation d'un groupe
- Anneau des polynômes et polynômes cyclotomiques
- Transformée de Fourier discrète

Réduction à l'octave et congruence modulo 12



$$a \sim b \pmod{n} \Leftrightarrow b - a = k \cdot n$$

$$a \sim b \pmod{12} \Leftrightarrow b - a = k \cdot 12$$



Relation d'équivalence :

- Réflexive
- Symétrique
- Transitive

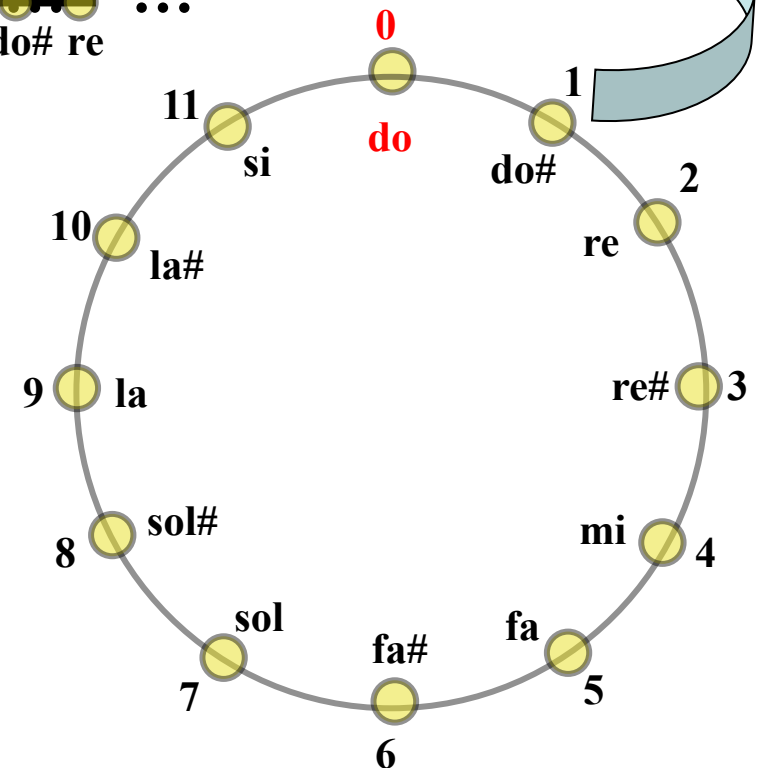
Congruence modulo 12 et structure de groupe (cyclique)



+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

... do do# re re# mi fa fa# sol sol# la la# si do do# re ...

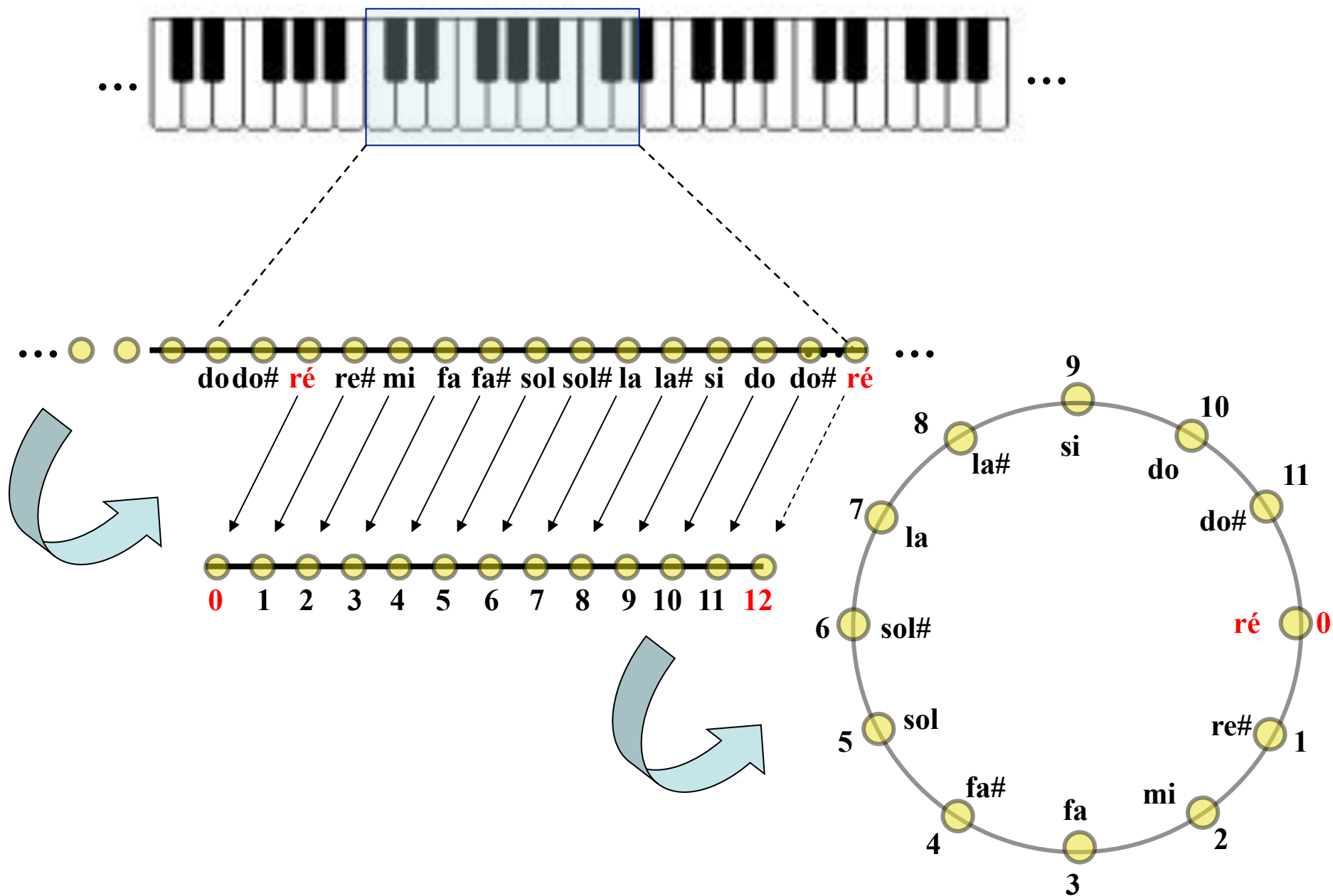
0 1 2 3 4 5 6 7 8 9 10 11 12



Structure de *groupe*

- Clôture
- Existence de l'élément neutre
- Existence de l'inverse
- Associativité

Principe du *do* mobile



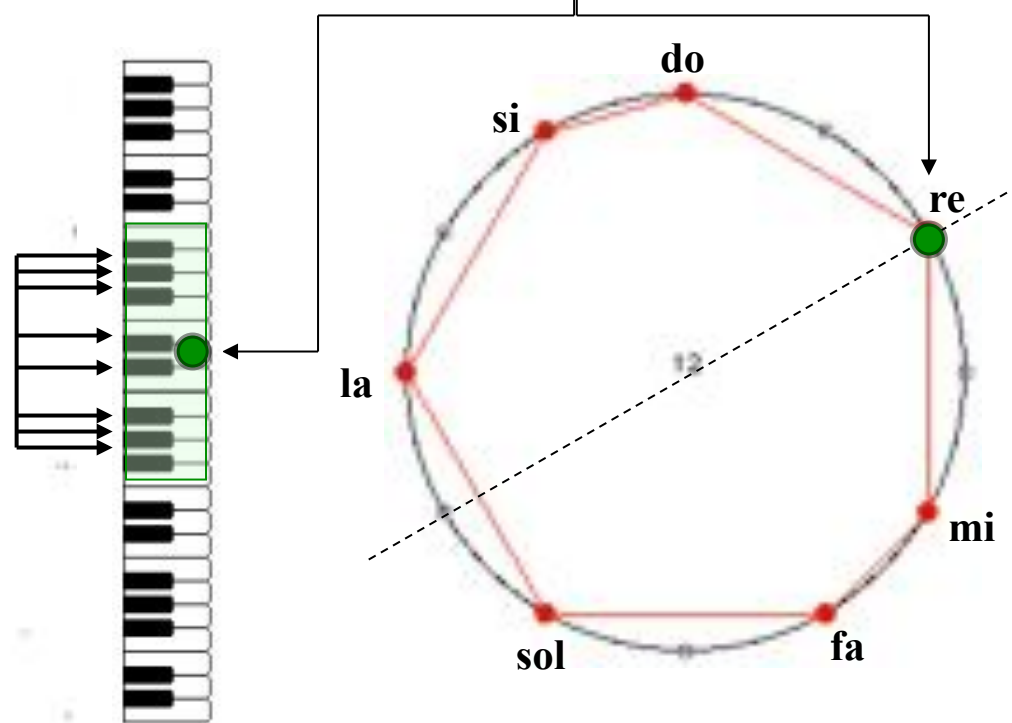
Un peu d'histoire...



Camille Durutte:

- *Technie, ou lois générales du système harmonique* (1855)
- *Résumé élémentaire de la Technie harmonique, et complément de cette Technie* (1876)

So	bb	Re	bb	Fa	Ut	Sol	Re	La	Mi	Si	Re	La
-15	-14	-3	-2	-1	0	+1	+2	+3	+14	+15		



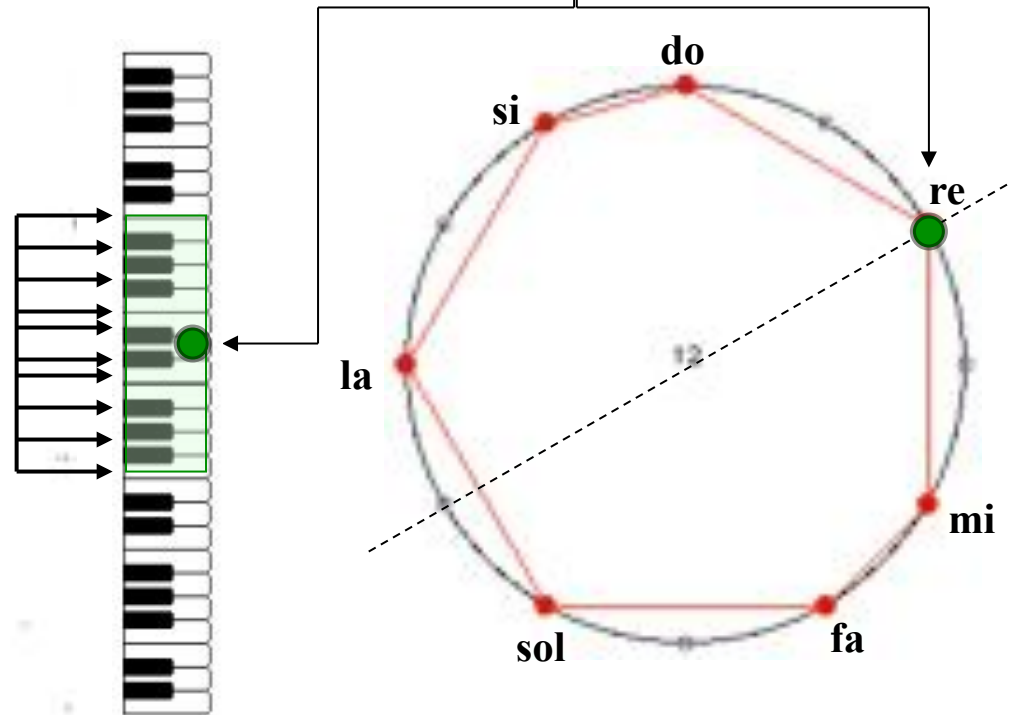
Un peu d'histoire...



Camille Durutte:

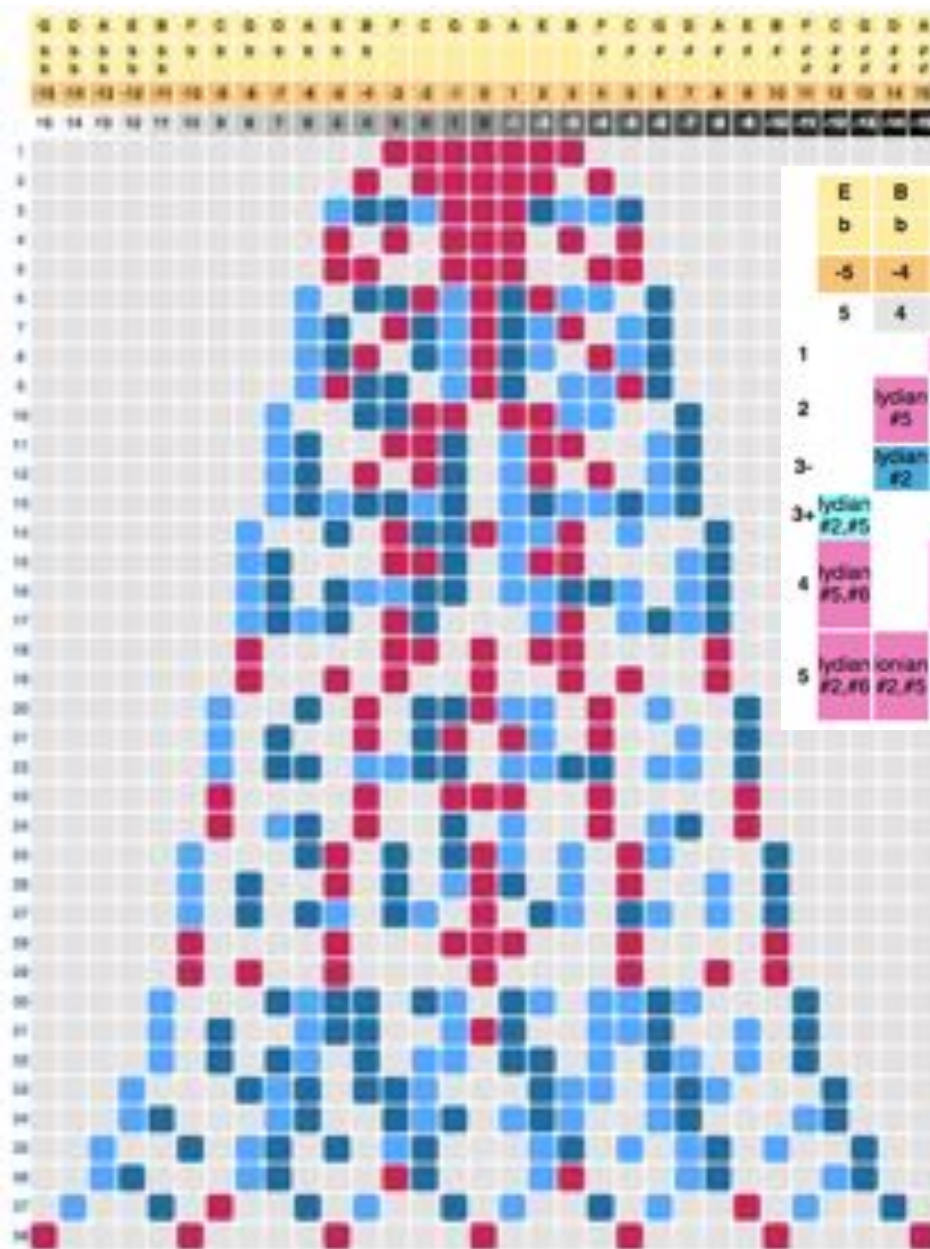
- *Technie, ou lois générales du système harmonique* (1855)
- *Résumé élémentaire de la Technie harmonique, et complément de cette Technie* (1876)

So	bb	Re	bb	Fa	Ut	Sol	Re	La	Mi	Si	Re	La
-15	-14	-3	-2	-1	0	+1	+2	+3	+14	+15		



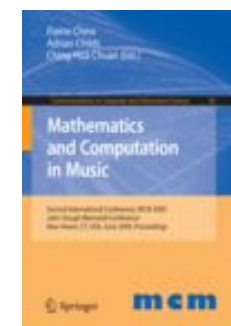
La cloche diatonique centrée autour du *ré* (P. Audétat & co.)

<http://www.cloche-diatonique.ch/>

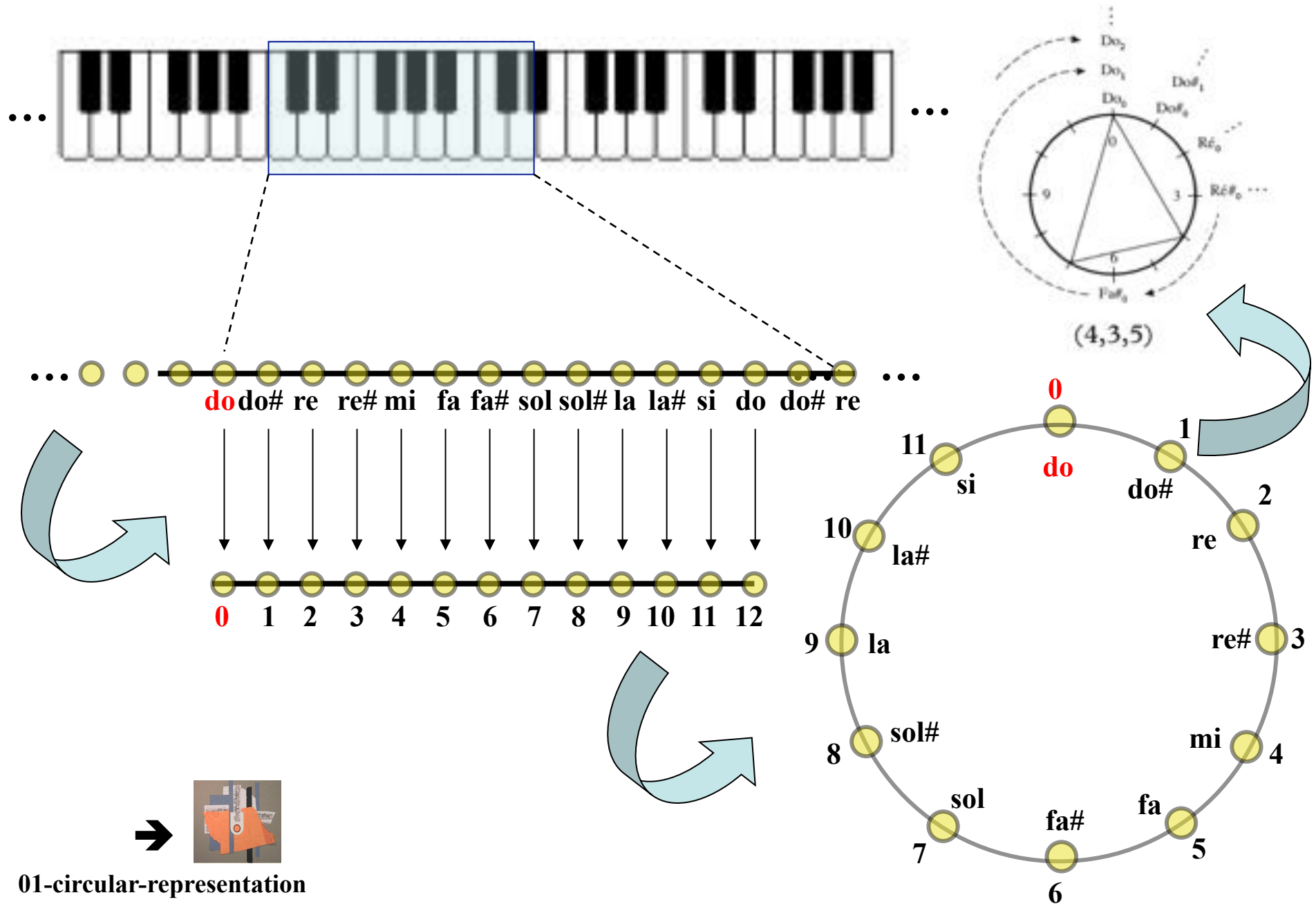


	E	B	F	C	G	D	A	E	B	F	C	
	b	b								#	#	
	-5	-4	-3	-2	-1	0	1	2	3	4	5	
	5	4	3	2	1	0	-1	-2	-3	-4	-5	
1			lydian	ionian	mixolydian	dorian	aeolian	phrygian	locrian			diatonic
2	lydian #5			lydian b7	ionian b3	mixolydian b5 or aeolian 3	phrygian 6	locrian 2		locrian b4		minor melodic
3		lydian #2	augmented ionian		dorian #4	aeolian 7	phrygian 3	locrian 6			locrian b4,bb7	minor harmonic
3+	lydian #2,#5			lydian b3	ionian b6	mixolydian b2	dorian b5		phrygian b4	locrian b6,7		major harmonic
4	lydian #5,#6		lydian #5,b7		lydian b6,b7	ionian b2,b3 or phrygian 6,7	locrian 2,3		locrian 2,b4		locrian bb3,b4	unisonic
5	lydian #2,#6	ionian #2,#5			aeolian #4,7	ionian b2,b6 or phrygian 3,7	mixolydian b2,b5			phrygian b4,bb7	locrian bb3,bb7	double harmonic

Junod, J., Audétat, P., Agon, C.,
 Andreatta, M., « A Generalisation of
 Diatonicism and the Discrete Fourier
 Transform as a Mean for Classifying and
 Characterising Musical Scales », Second
 International Conference MCM 2009,
 vol. 38, New Haven, 2009, pp. 166-179

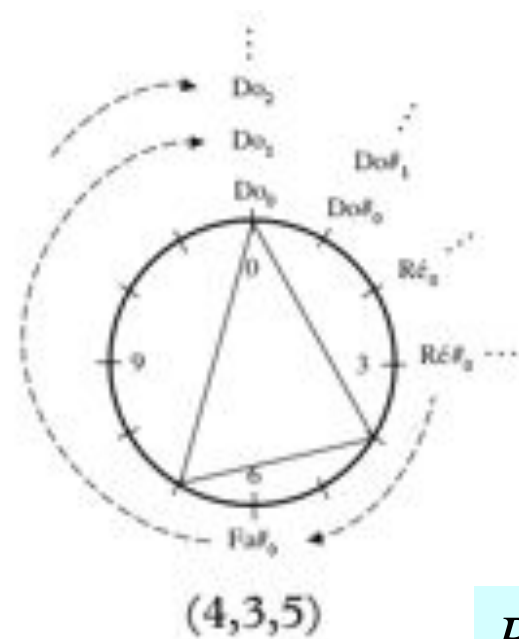


Premiers invariants algébriques : la structure intervallique

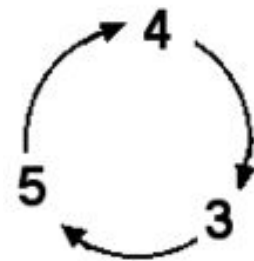


01-circular-representation

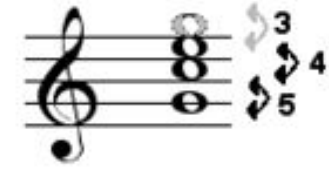
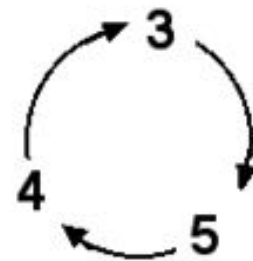
Structure intervallique et renversements d'accords



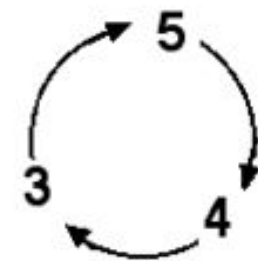
(4 3 5)



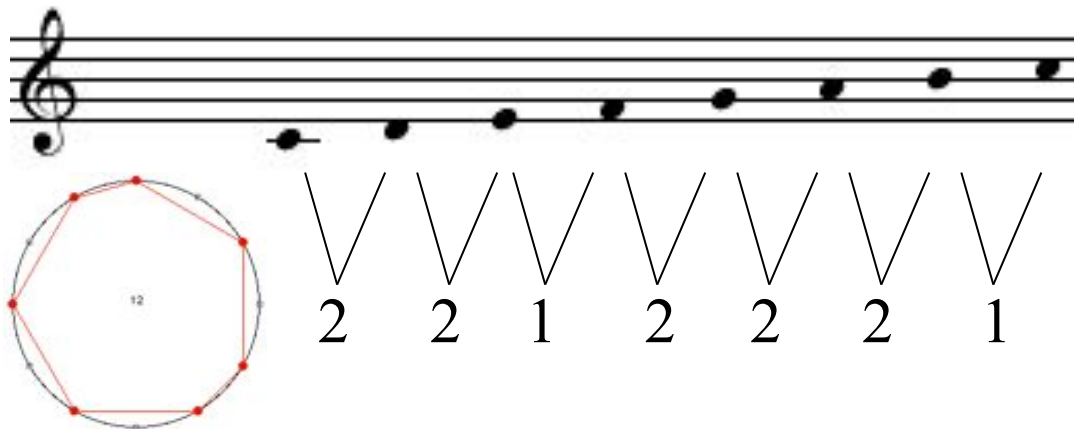
(3 5 4)



(5 4 3)

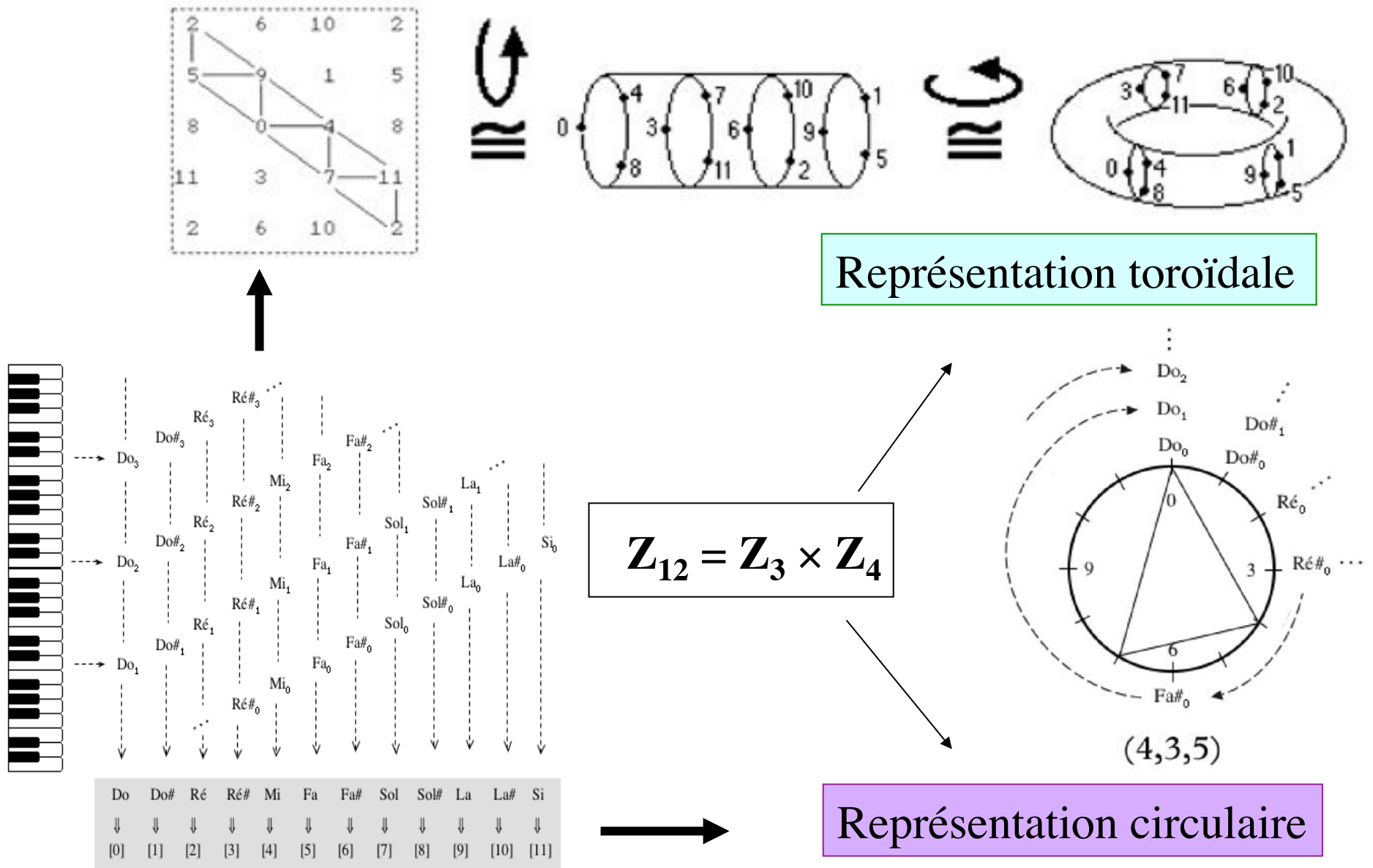


Renversements = permutations circulaires de la structure intervallique

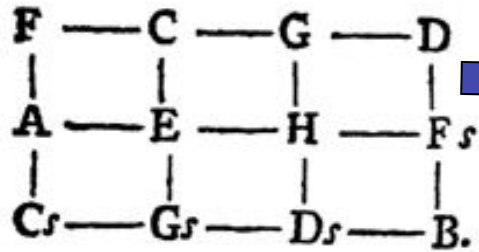


	E	B	F	C	G	D	A	E	B	F	C	
	b	b								#	#	
	-5	-4	-3	-2	-1	0	1	2	3	4	5	
	5	4	3	2	1	0	-1	-2	-3	-4	-5	
1			lydian	ionian	mixolydian	dorian	aeolian	phrygian	locrian		diatonic	
2		lydian #5		lydian b7	ionian b3	mixolydian b6 or aeolian 3	phrygian 6	locrian 2		locrian b4	minor melodic	
3		lydian augmented	ionian		dorian #4	aeolian 7	phrygian 3	locrian 5			locrian b4,bb7	minor harmonic
3#	lydian #2,#5			lydian b3	ionian b6	mixolydian b2	dorian b5		phrygian b4	locrian bb7		major harmonic
4	lydian #5,#6		lydian #5,b7		lydian b6,b7	ionian b2,b3 or phrygian 6,7	locrian 2,3		locrian 2,b4		locrian bb3,b4	untonic
5	lydian #2,#6	ionian #2,#5			aeolian #4,7	ionian b2,b6 or phrygian 5,7	mixolydian b2,b5			phrygian b4,bb7	locrian bb3,bb7	double harmonic

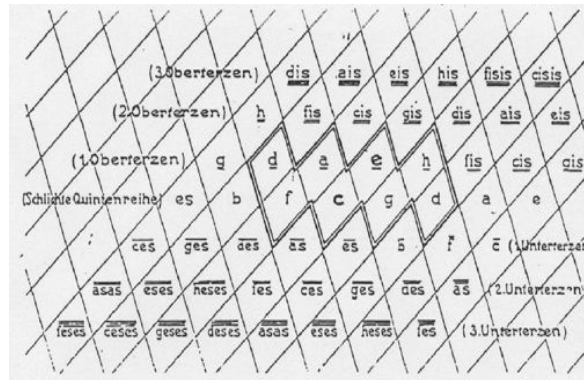
Equivalence algébrique entre représentations géométriques



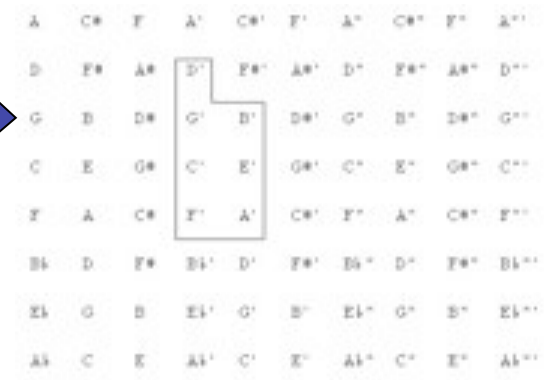
Du *Speculum musicum* d'Euler au Tonnetz de Riemann



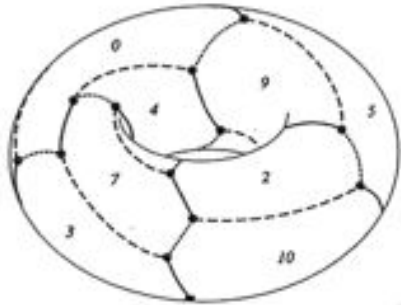
Euler : *Speculum musicum*, 1773



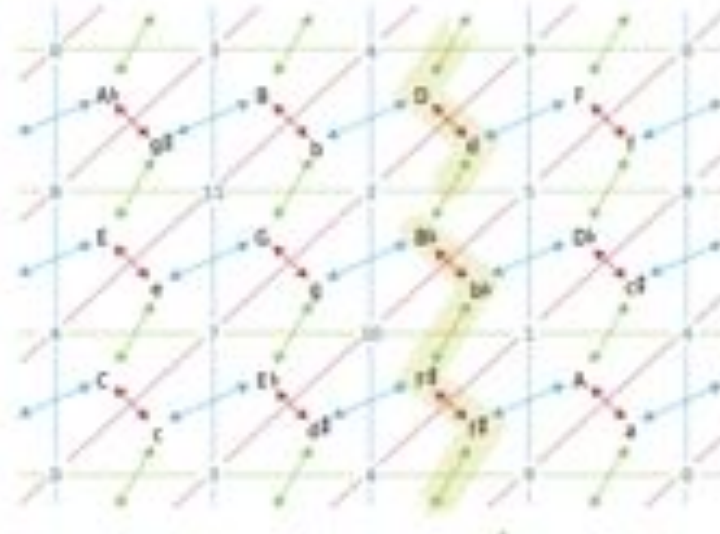
Hugo Riemann : « Ideen zu einer Lehre von den Tonvorstellung », 1914



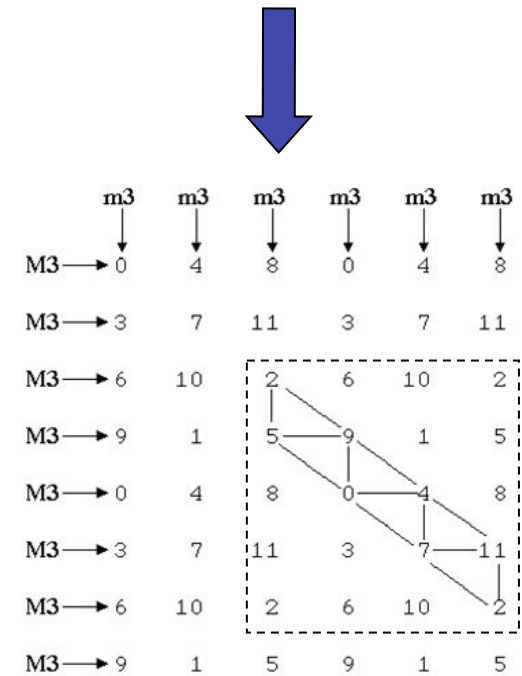
Longuet-Higgins (1962)



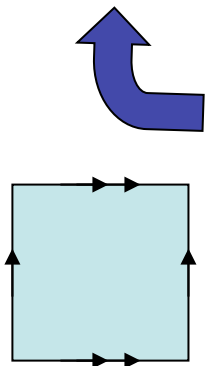
Douthett & Steinbach, *JMT*, 1998



J. Hook, « Exploring Musical Space », *Science*, 2006



Balzano (1980)



Vers l'émergence de la notion de groupe en musique

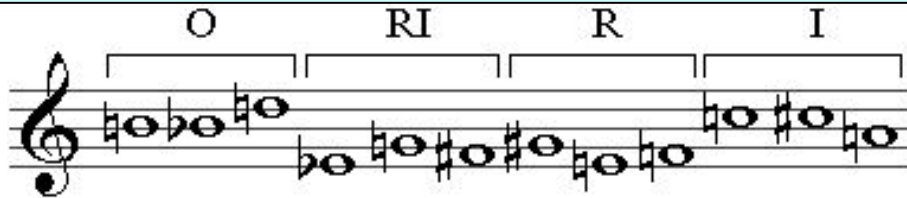
Ernst Krenek et l'approche axiomatique en musique

- *The Relativity of Scientific Systems*
- *The Significance of Axioms*
- *Axioms in music*
- *Musical Theory and Musical Practice*

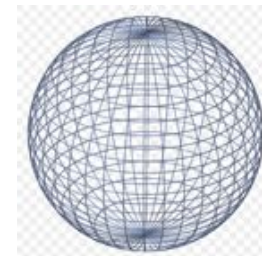
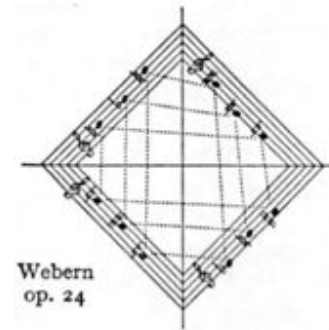
Ernst Krenek : *Über Neue Musik*, 1937
(Engl. Transl. *Music here and now*, 1939)



Physicists and mathematicians are far in advance of musicians in realizing that their respective sciences do not serve to establish a concept of the universe conforming to an objectively existent nature



As the study of axioms eliminates the idea that axioms are something absolute, conceiving them instead as free propositions of the human mind, just so would this musical theory free us from the concept of major/minor tonality [...] as an irrevocable law of nature.



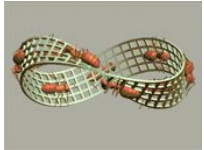
L'approche axiomatique en mathématique

David Hilbert: les fondements axiomatiques de la géométrie et le rôle de l'intuition

*In order to be constructed in a right way, geometry [...] only needs few simple principles. These principles are called the **axioms** of the geometry. [...] This study (of the axioms) goes back to the **logical analysis of our spatial intuition** (Grundlage der Geometrie, 1899).*



*At the moment there are two tendencies in mathematics. From one side, the tendency toward abstraction aims at 'crystallizing' the logical relations inside of a study object and at organizing this material in a systematic way. But there is also a tendency towards the **intuitive understanding** which aims at understanding the **concret meaning of their relations** (Anschauliche Geometrie, 1932)*



Structures d'orbifolds en analyse musicale

$$T^2 = \mathbf{R}/12\mathbf{Z} \times \mathbf{R}/12\mathbf{Z} \longrightarrow T^2 / S_2$$

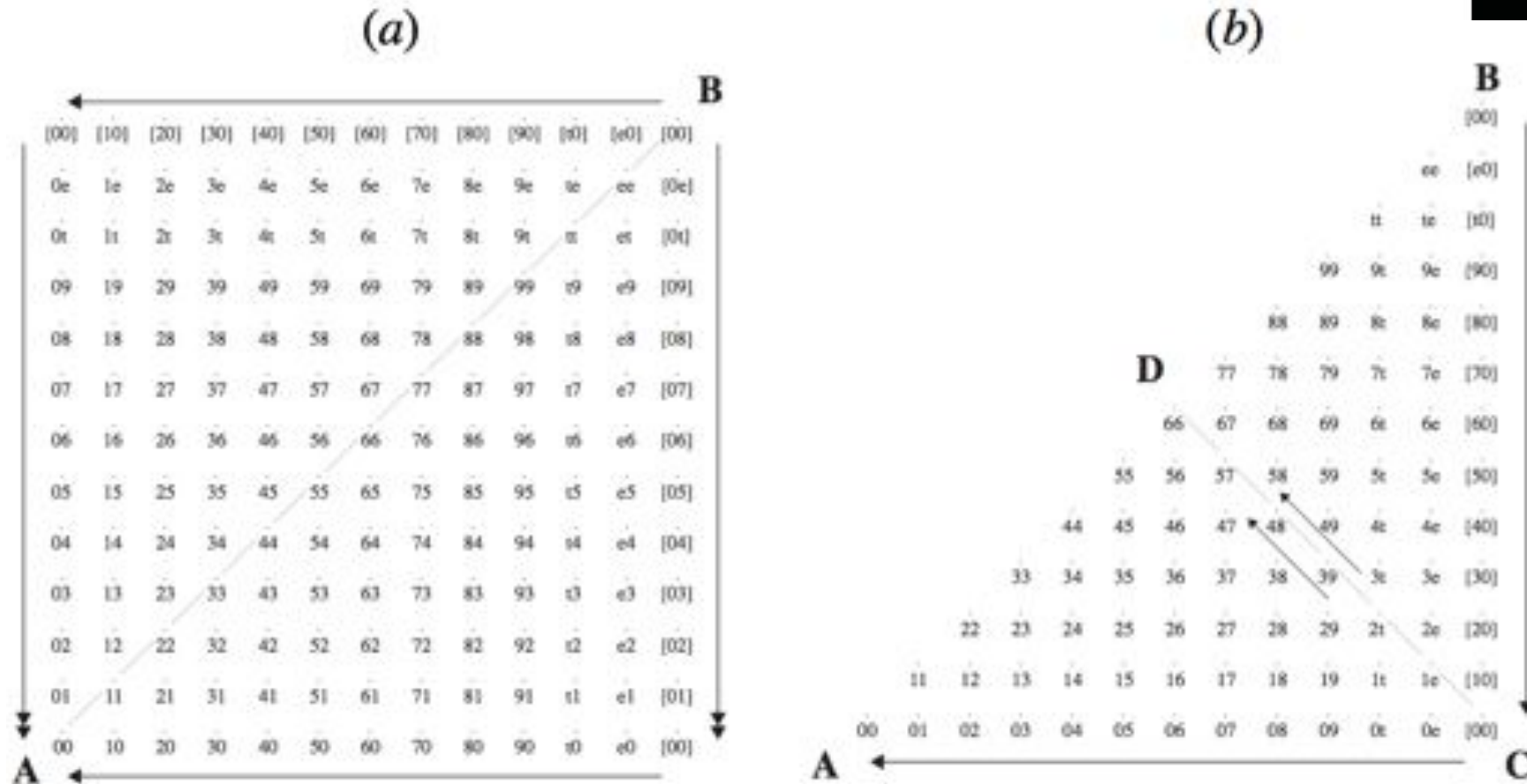
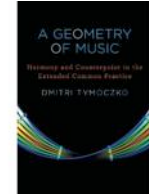
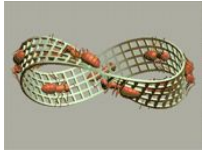


Figure S9. (a) The space of ordered two-note chords of pitch-classes is a 2-torus. To identify points (x, y) and (y, x) , we need to fold the torus along the AB diagonal. The resulting figure, shown in (b), is a triangle with two of its sides identified. This is a Möbius strip. To see why, cut figure (b) along the line CD and glue AC to CB. (To make this identification in Euclidean 3-space, you will need to turn over one of the pieces of paper.) The result is a square with opposite sides identified, as in Figure 2 of the main paper. Dmitri Tymoczko : « The Geometry of Musical Chords », *Science*, 313, 2006



Structures d'orbifolds en analyse musicale

$$T^2 = \mathbf{R}/12\mathbf{Z} \times \mathbf{R}/12\mathbf{Z} \longrightarrow T^2 / S_2$$

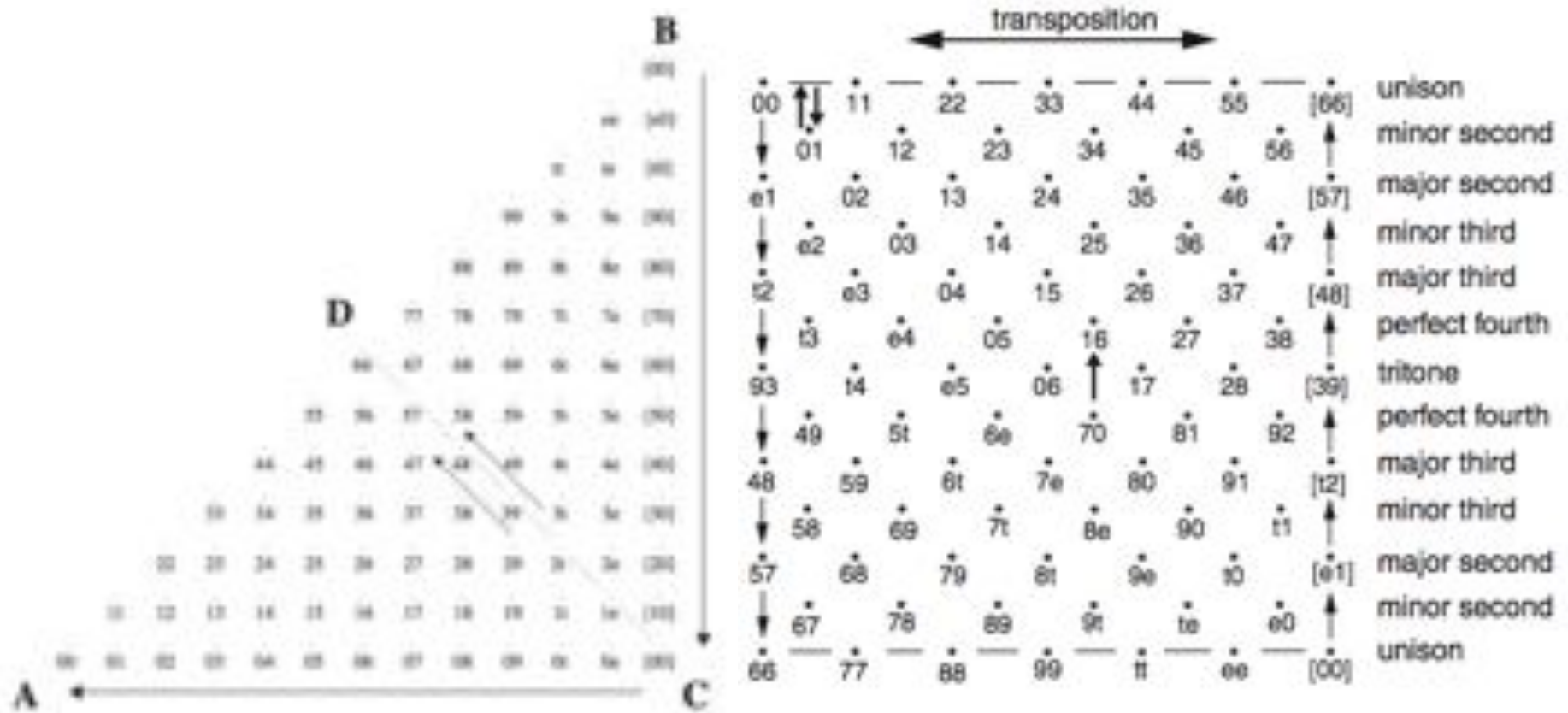
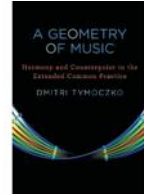
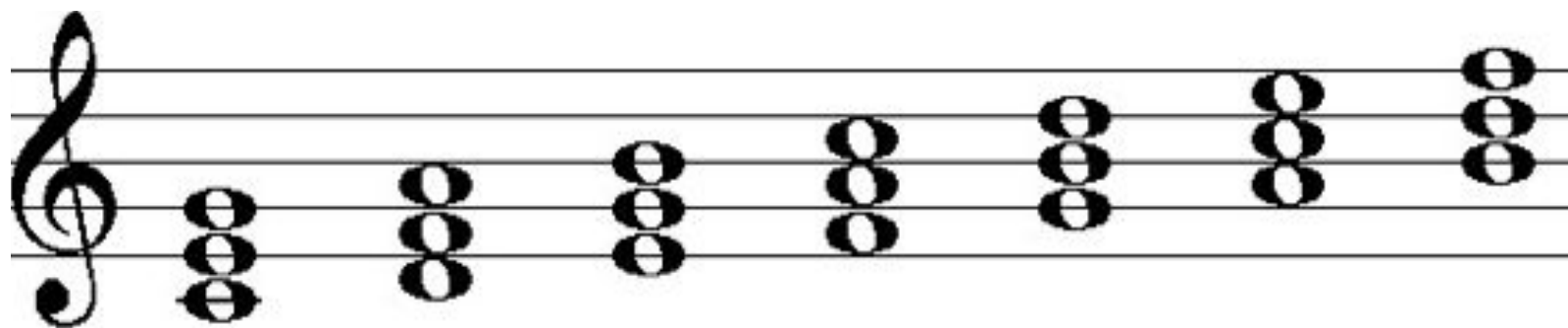


Figure S9. (a) *The space of ordered two-note chords of pitch-classes is a 2-torus. To identify points (x, y) and (y, x) , we need to fold the torus along the AB diagonal. The resulting figure, shown in (b), is a triangle with two of its sides identified. This is a Möbius strip. To see why, cut figure (b) along the line CD and glue AC to CB. (To make this identification in Euclidean 3-space, you will need to turn over one of the pieces of paper.) The result is a square with opposite sides identified, as in Figure 2 of the main paper.* Dmitri Tymoczko :« The Geometry of Musical Chords », *Science*, 313, 2006



I

II

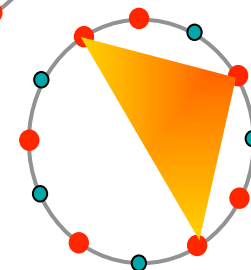
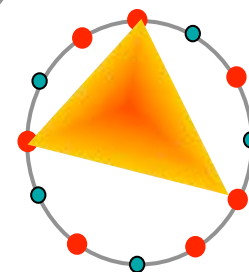
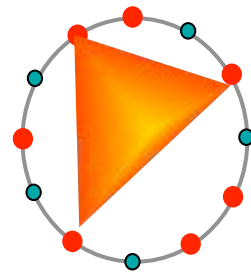
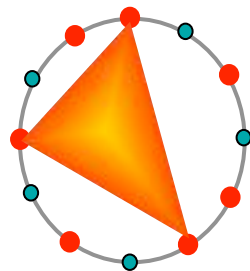
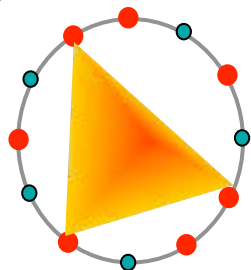
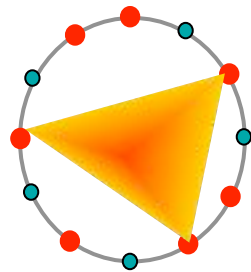
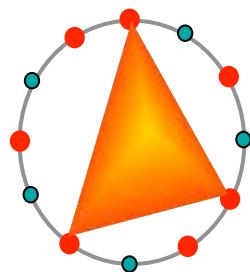
III

IV

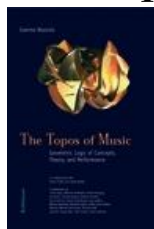
V

VI

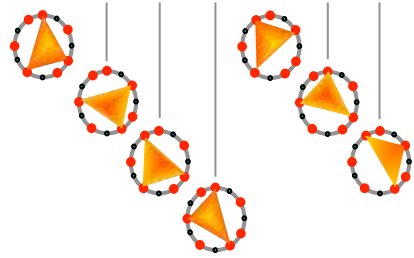
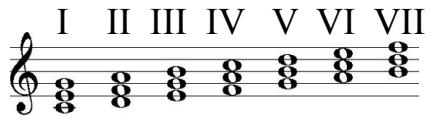
VII



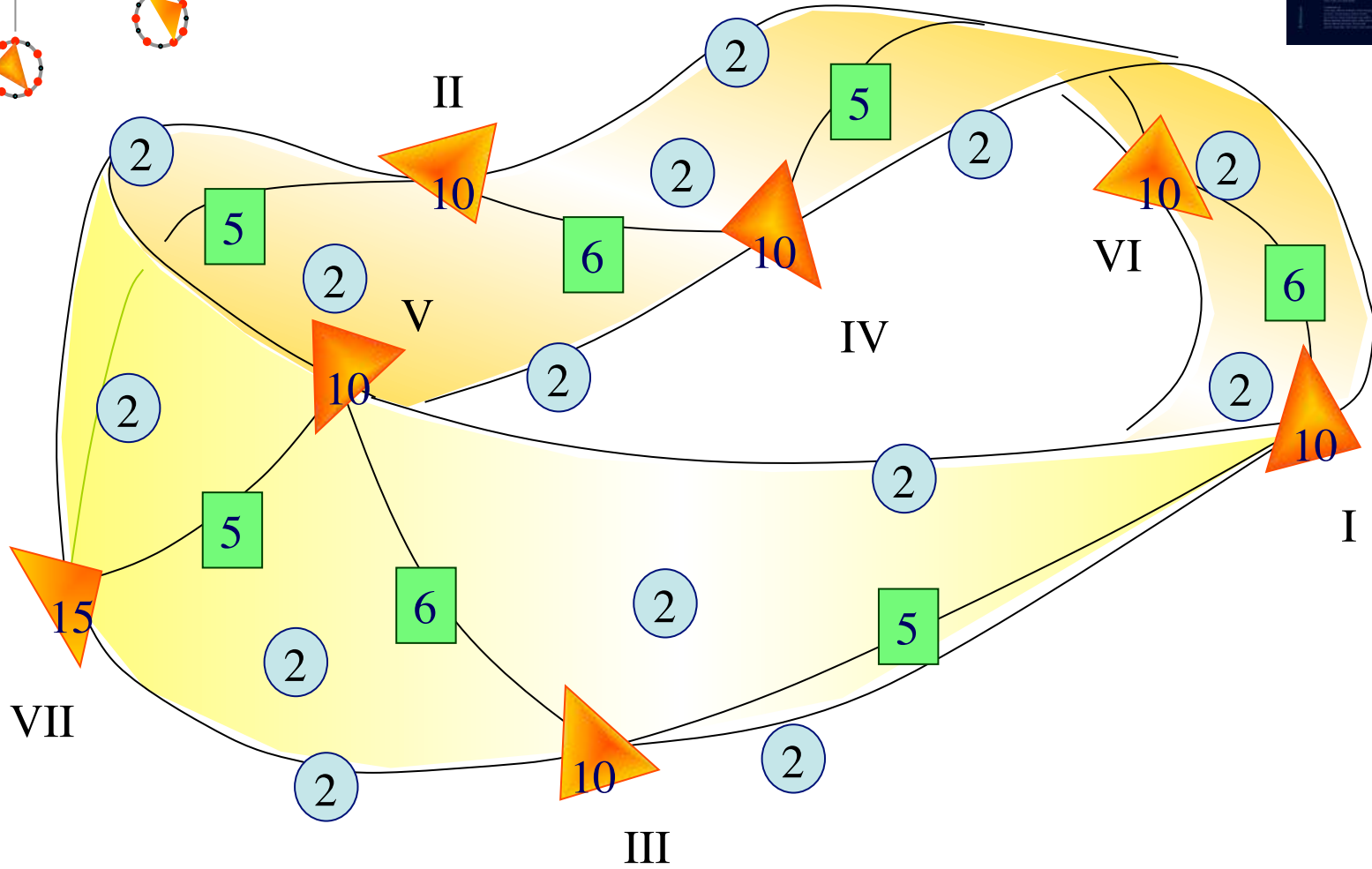
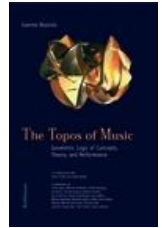
Un atlas pour la gamme diatonique...



G. Mazzola, *The Topos of Music*



G. Mazzola, *The Topos of Music*



...et le nerf topologique associé

Nerf topologique et théorie de la modulation

G. Mazzola, *La vérité du beau dans la musique*, Collection « Musique/Sciences », IRCAM/Delatour, 2007.

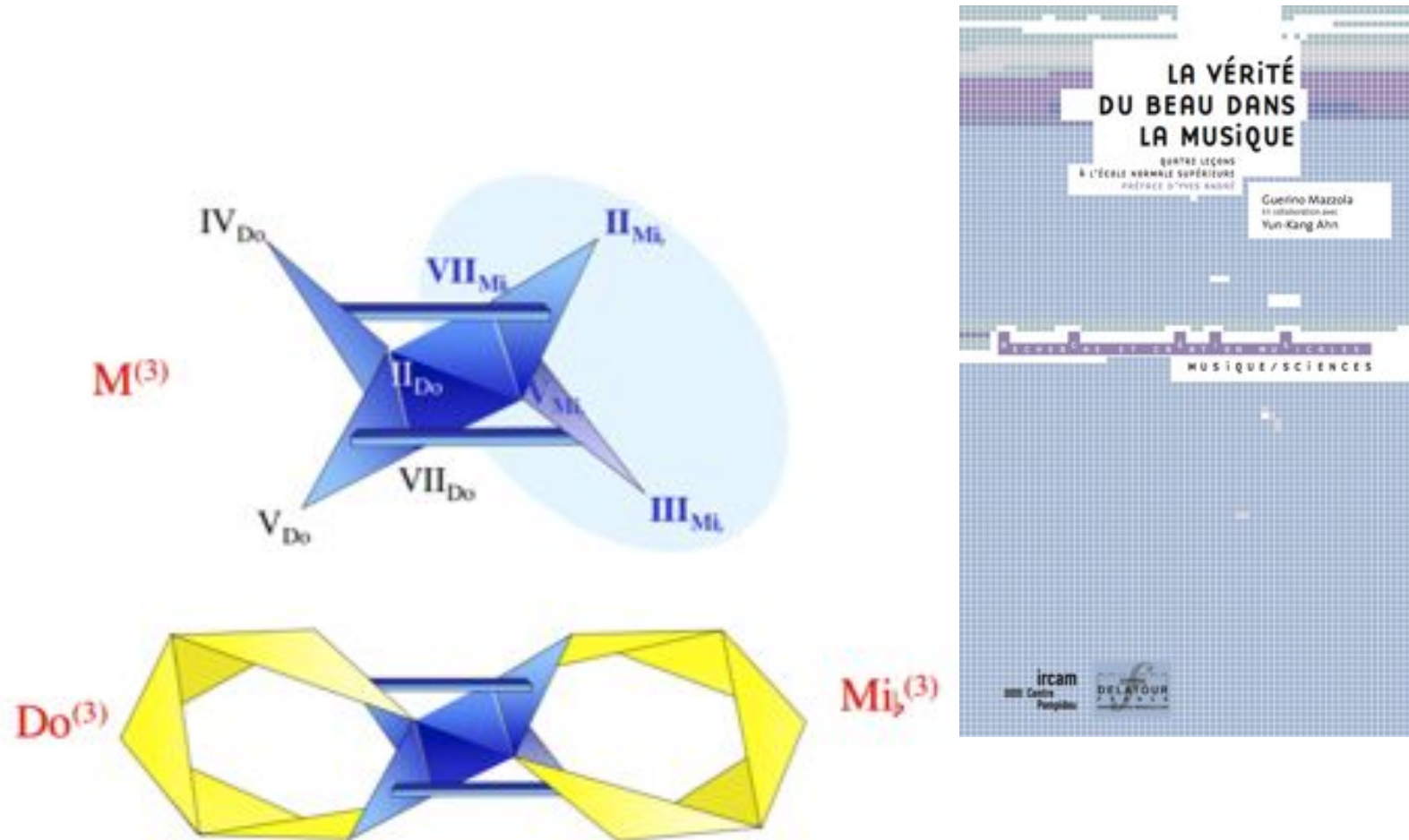
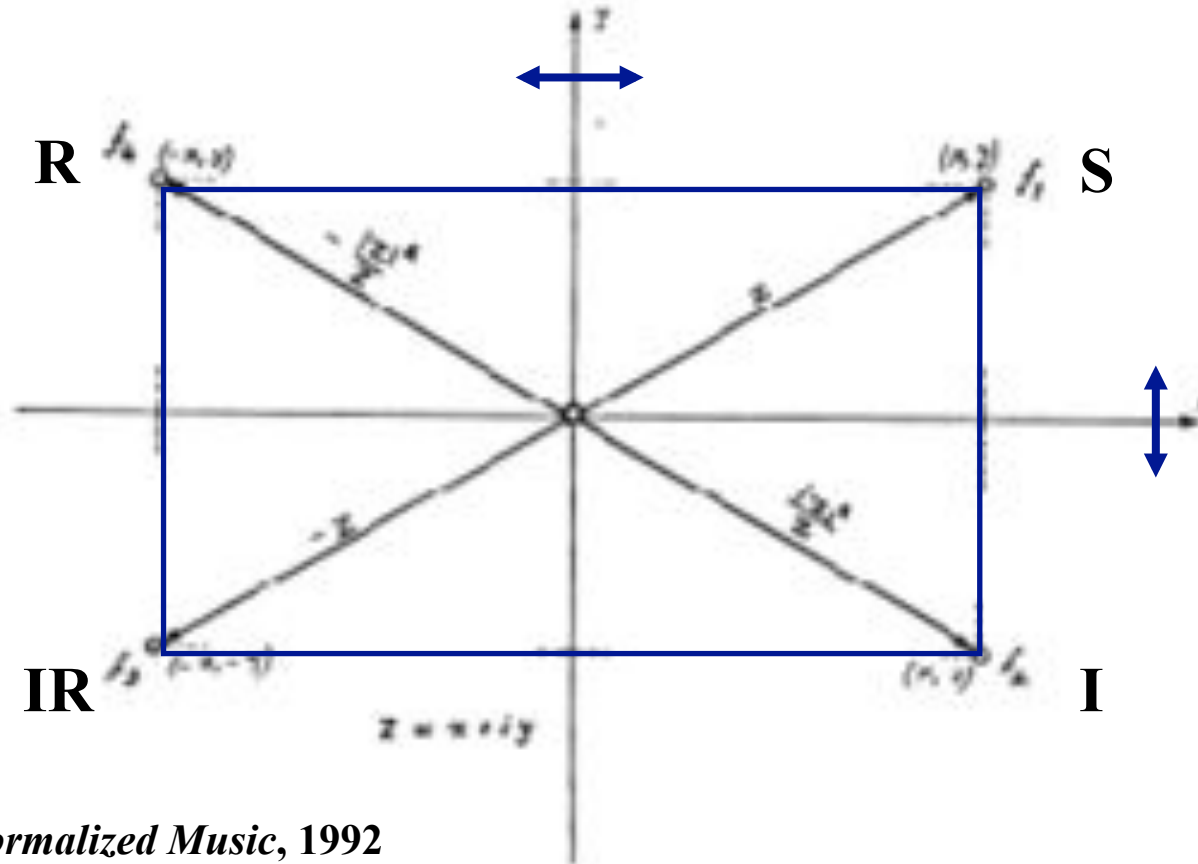
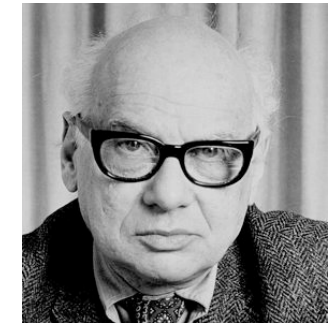


FIG. 8.6 – Le quantum de modulation de la tonalité $Do^{(3)}$ dans la tonalité $Mi^{(3)}$ dans sa représentation topologique via le nerf. Le nerf est de dimension cinq (six sommets en position générale), ce qui est symbolisé par les barres horizontales reliant six degrés chacune. En haut à droite sont représentés les pivots II, III, V, VII .

Opérations dodécaphoniques et structures algébriques

Série d'origine
Inversion
Rétrogradation
Rétrogradation inverse

	S	I	R	RI
S	S	I	R	RI
I	I	S	RI	R
R	R	RI	S	I
RI	RI	R	I	S



Iannis Xenakis, *Formalized Music*, 1992

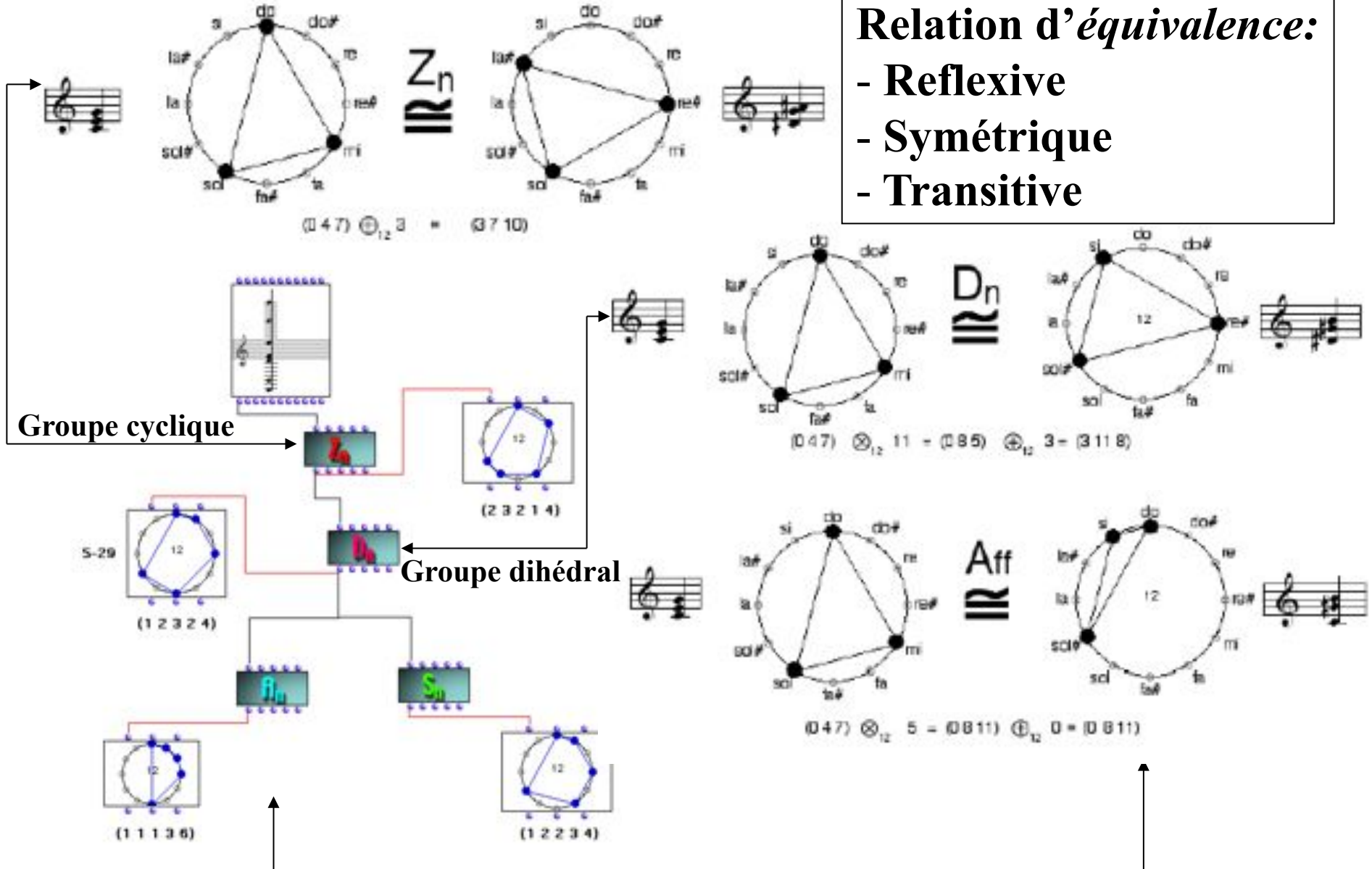


Felix Klein

Les groupes comme "paradigmes"

Relation d'équivalence:

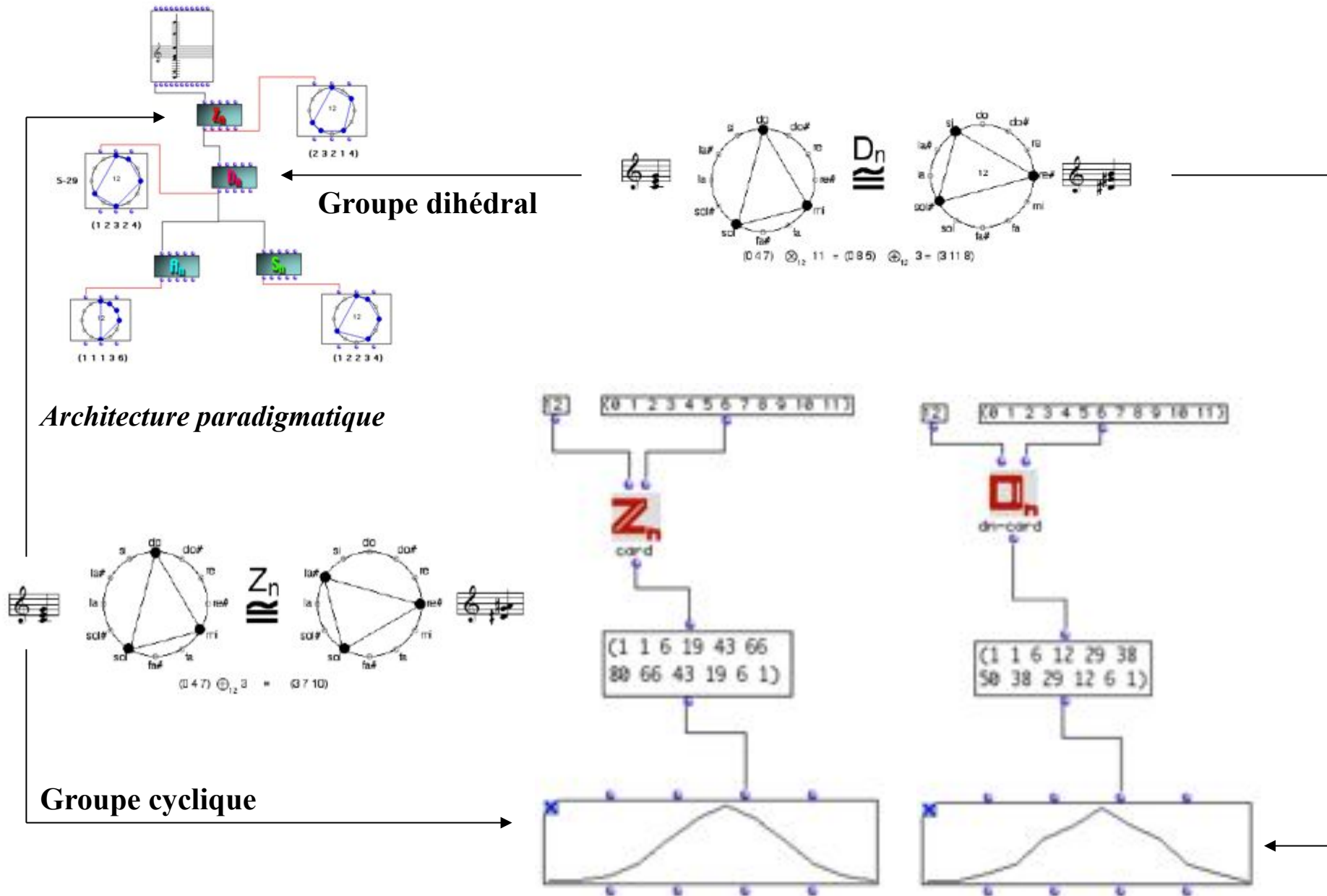
- Reflexive
- Symétrique
- Transitive



Architecture paradigmatique

Groupe Affine

Classes d'équivalence d'accords et groupes de transformations

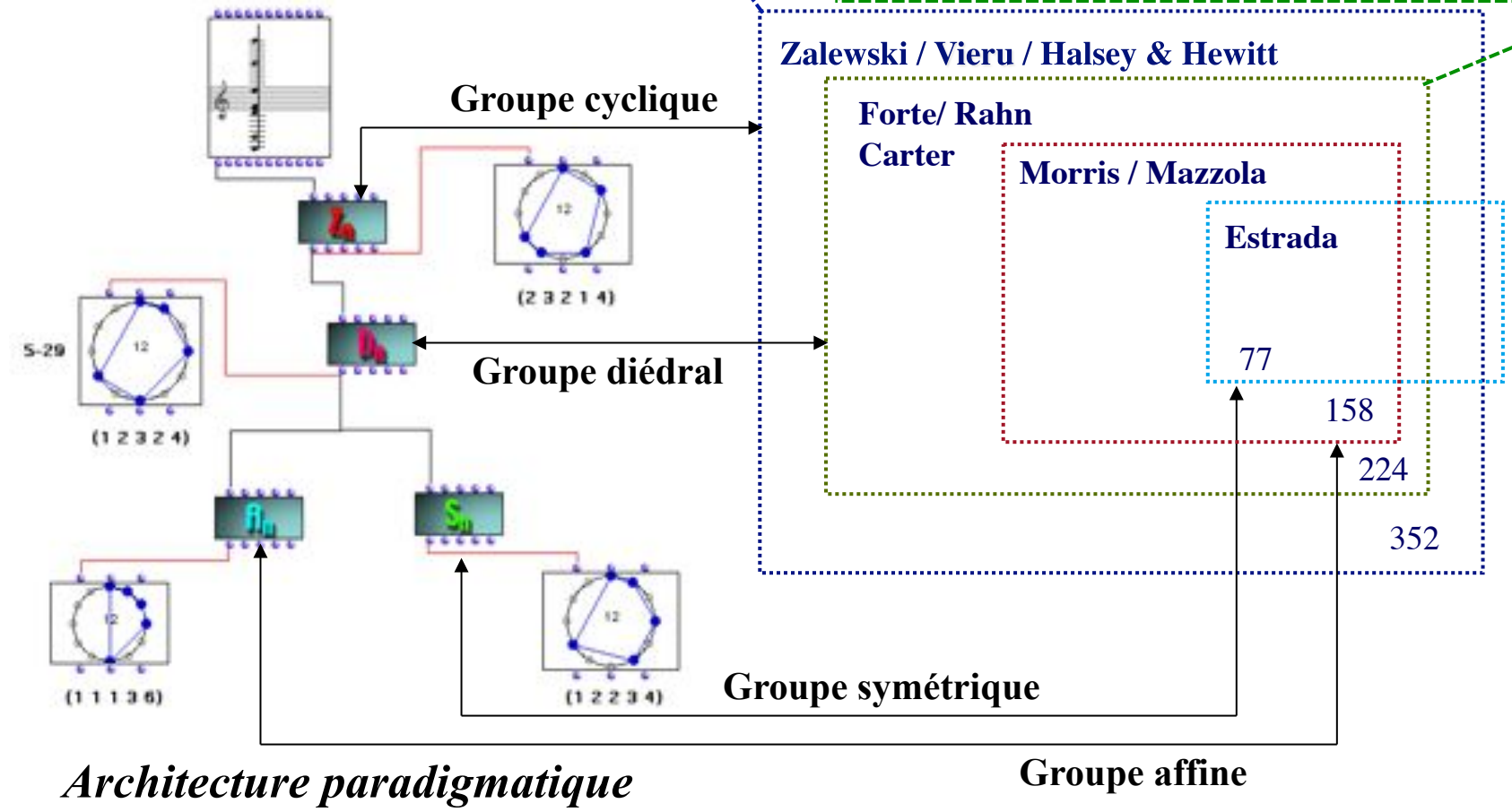


Classification paradigmatic of musical structures

$$\# \text{ of } k\text{-chords} = \frac{1}{n} \sum_{j|(n,k)} \phi(j) \binom{n/j}{k/j} = \frac{1}{n} \Phi_n(k)$$

$\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$
 $n \mapsto \text{card}(\{m \in \mathbb{N}^* \mid m \leq n \text{ et } m \text{ premier avec } n\})$

$$\# \text{ of } k\text{-chords} = \begin{cases} \frac{1}{2n} \left[\Phi_n(k) + n \binom{(n-1)/2}{[k/2]} \right], & \text{if } n \text{ is odd,} \\ \frac{1}{2n} \left[\Phi_n(k) + n \binom{n/2}{k/2} \right], & \text{if } n \text{ is even and } k \text{ is even,} \\ \frac{1}{2n} \left[\Phi_n(k) + n \binom{(n/2)-1}{[k/2]} \right], & \text{if } n \text{ is even and } k \text{ is odd.} \end{cases}$$



Architecture paradigmatic

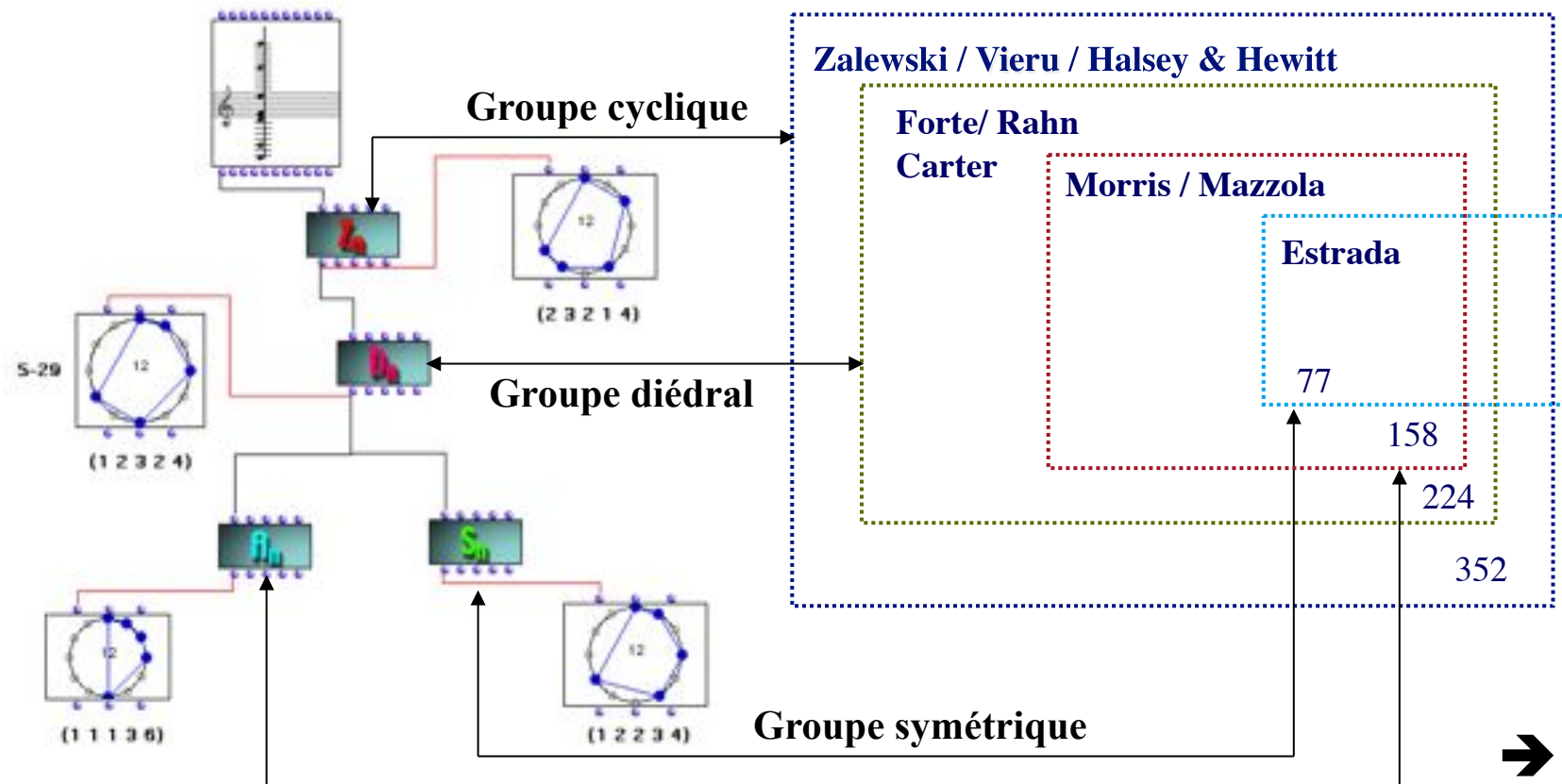
Groupe affine

- D. Halsey & E. Hewitt: « Eine gruppentheoretische Methode in der Musik-theorie », *Jahresber. Der Dt. Math.-Vereinigung*, 80, 1978.
- D. Reiner: « Enumeration in Music Theory », *Amer. Math. Month.* 92:51-54, 1985
- R.C. Read: « Combinatorial problems in the theory of music », *Discrete Math.*, 1997
- H. Friepertinger: « Enumeration of mosaics », *Discrete Math.*, 1999

Classification paradigmatic des structures musicales

$G \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
D_{12}	1	6	12	29	38	50	38	29	12	6	1	1
$\text{Aff}_1(Z_{12})$	1	5	9	21	25	34	25	21	9	5	1	1

Set Theory



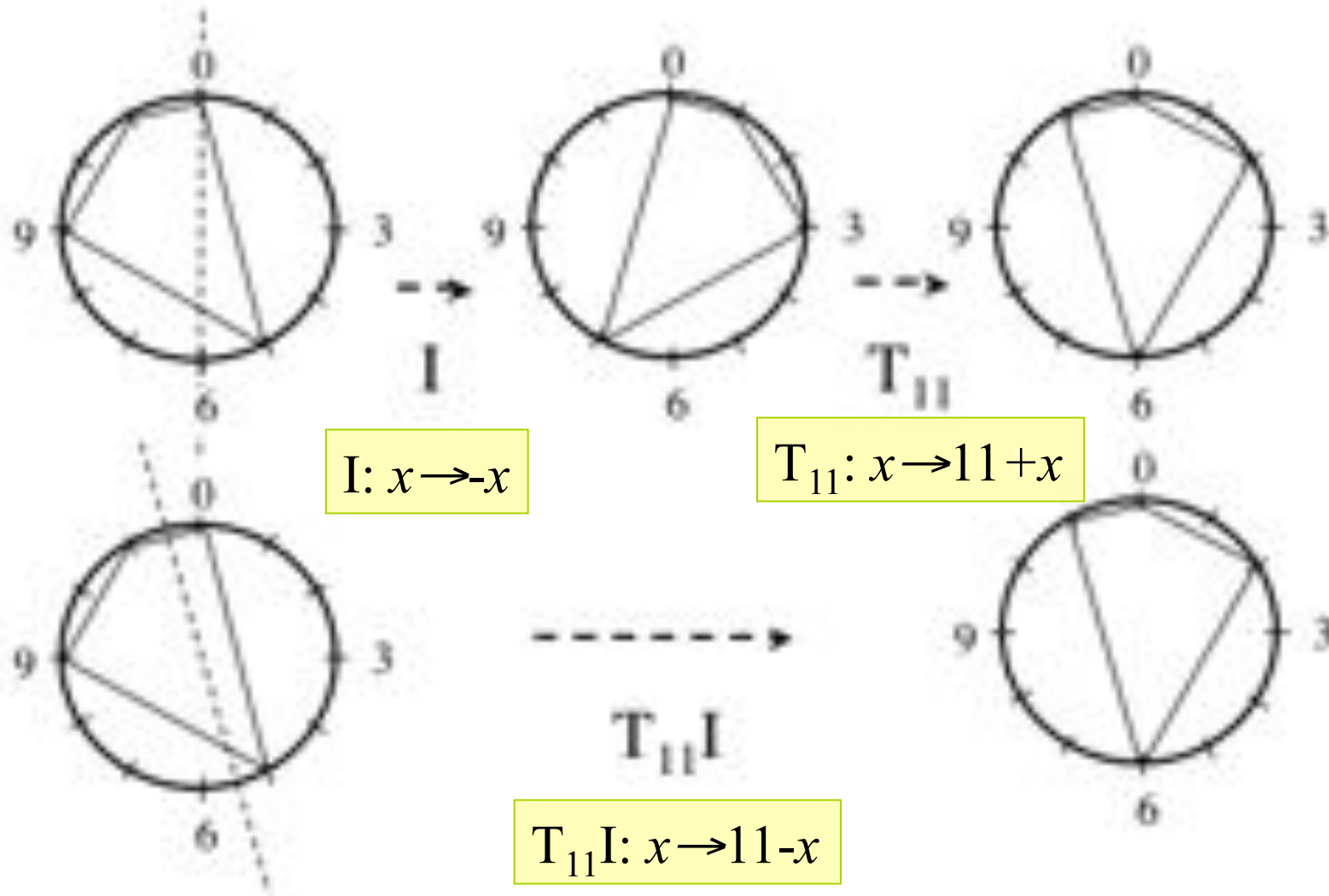
Architecture paradigmatique

Groupe affine



02-paradigms

La Set Theory: équivalence modulo transposition/inversion

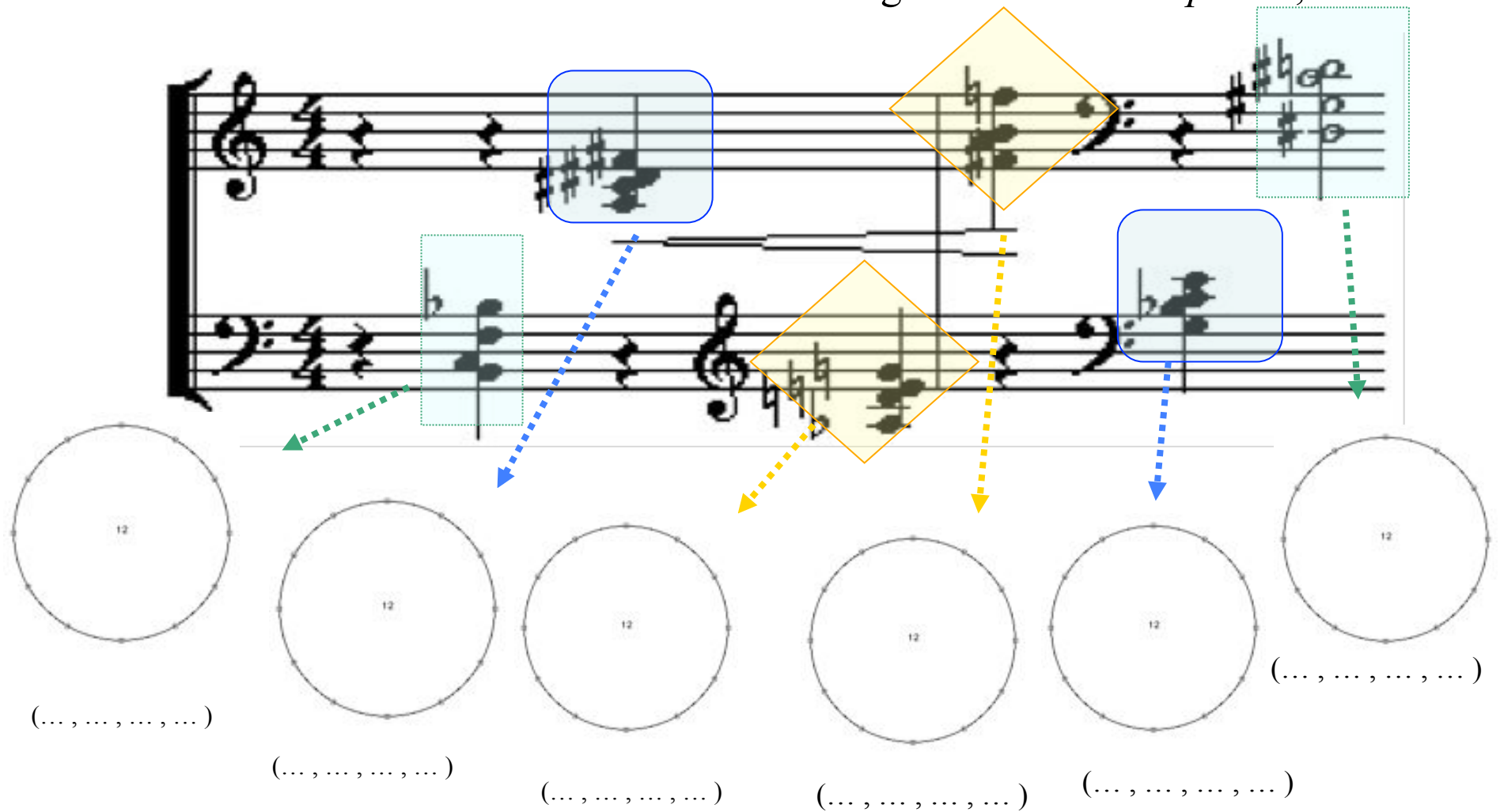


$$\{0, 5, 9, 11\} \longrightarrow \{11, 6, 3, 0\}$$

L'analyse formalisée ou les entités formelles en musique

André Riotte e Marcel Mesnage

A. Schoenberg : *Klavierstück Op. 33a*, 1929



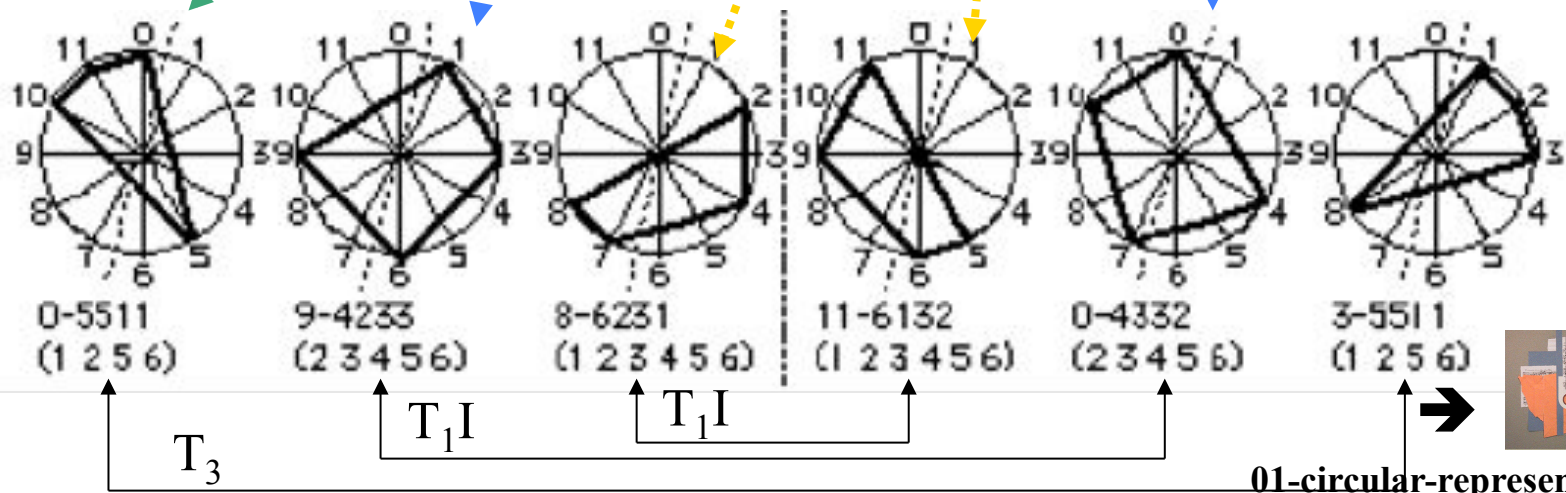
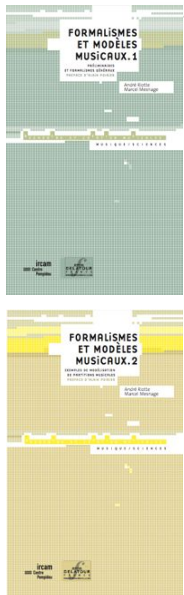


L'analyse formalisée ou les entités formelles en musique

André Riotte & Marcel Mesnage



A. Schoenberg : *Klavierstück Op. 33a*, 1929



01-circular-representation



Exercice : retrouver les symétries dans une série (I)

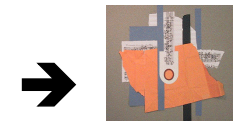
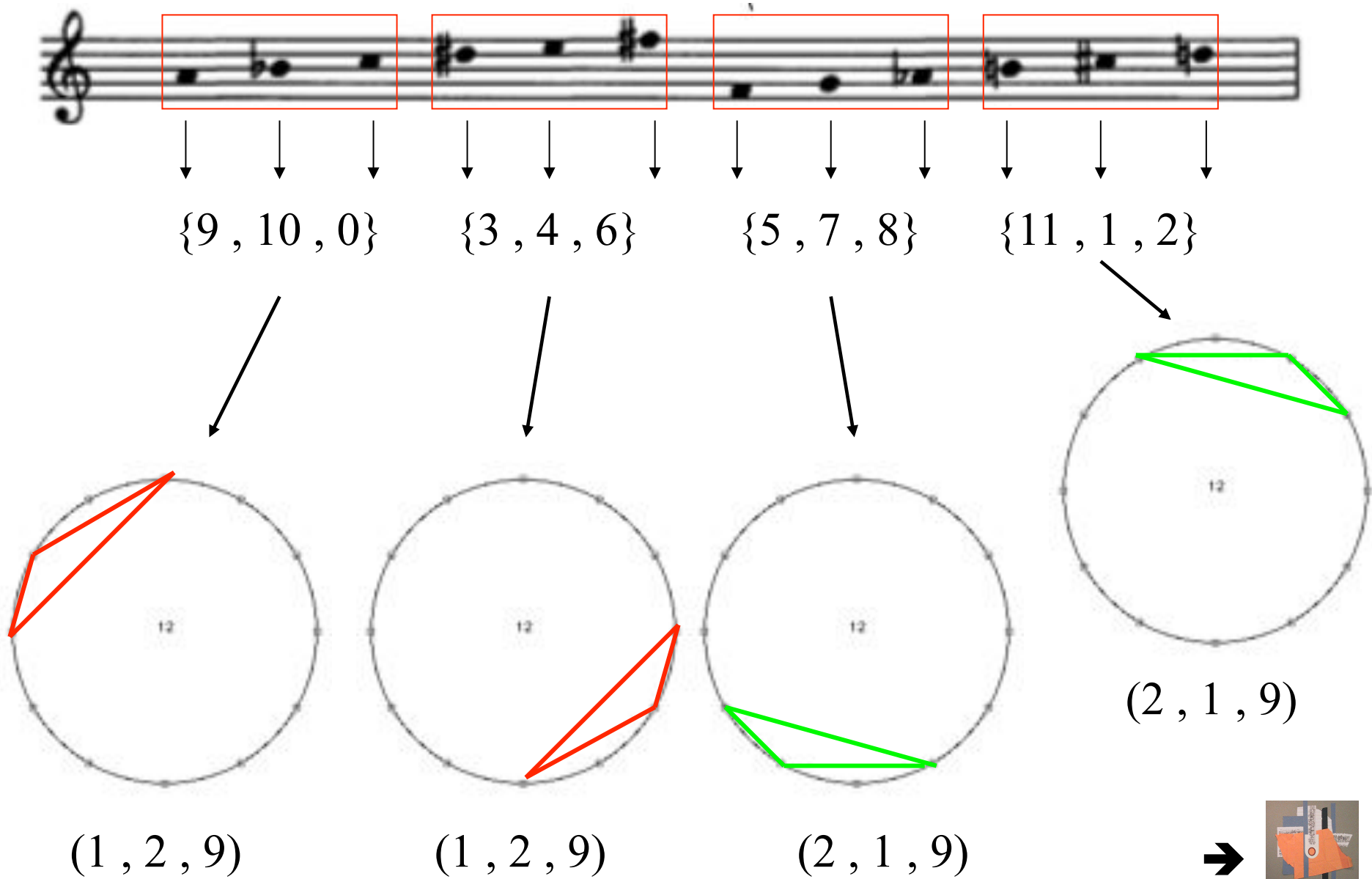
Schoenberg: Serenade Op.24, Mouvement 5

Diagram showing a musical staff with four groups of notes highlighted by red boxes. Below each group are three downward-pointing arrows and a set of three ellipses in curly braces, representing a sequence of notes.

Diagram showing four circles, each with 12 dots on its circumference and the number 12 in the center. Arrows point from the boxes in the previous block to these circles. Below each circle is a set of three ellipses in parentheses, representing a sequence of notes.

Exercice : retrouver les symétries dans une série (I)

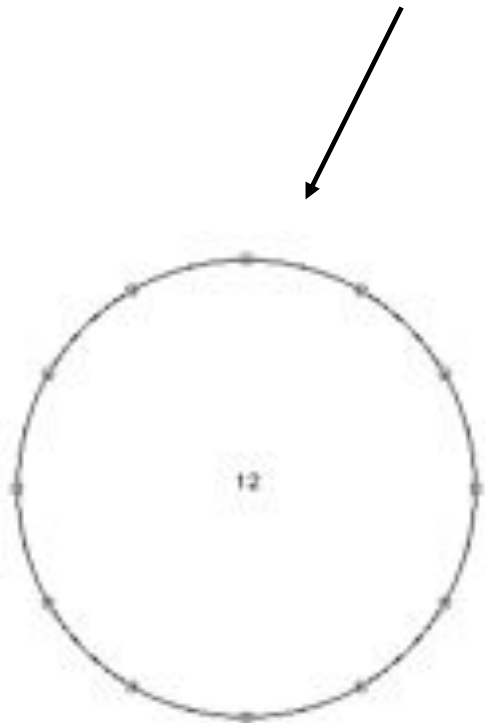
Schoenberg: Serenade Op.24, Mouvement 5



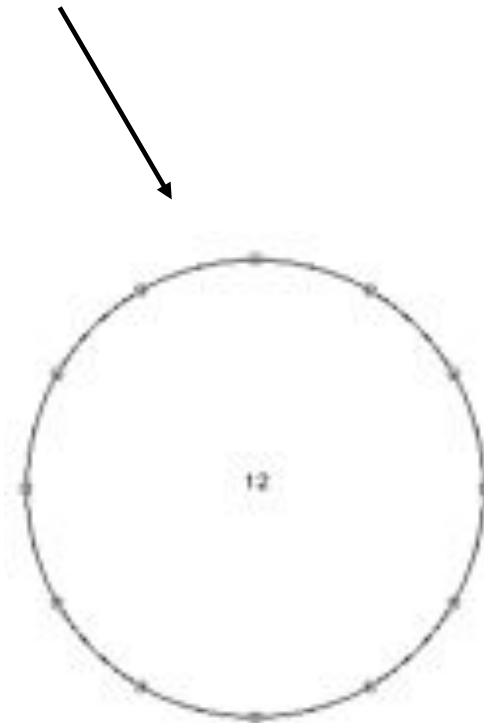
Exercice : retrouver les symétries dans une série (II)

Schoenberg: Serenade Op.24, Mouvement 5

The image shows a musical staff with a treble clef. Two segments of the series are highlighted with red boxes. The first box covers six notes, and the second box covers six notes. Below each box, a vertical arrow points to a set of six ellipses, representing the series of notes.



(... , ... , ... , ... , ... , ...)

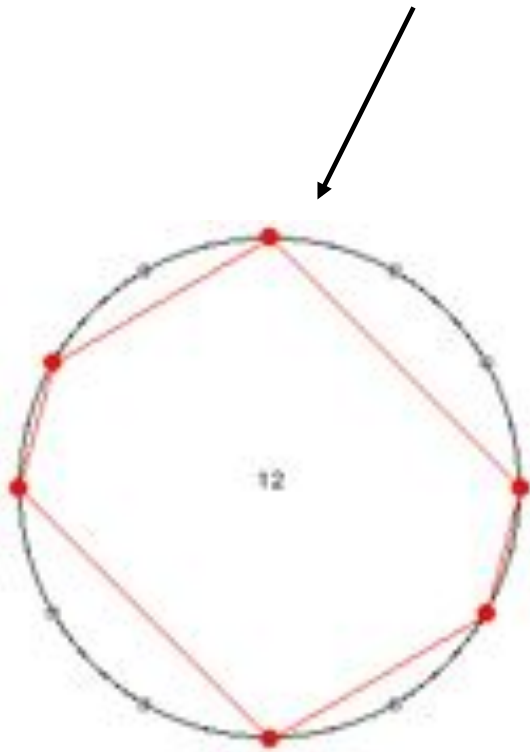


(... , ... , ... , ... , ... , ...)

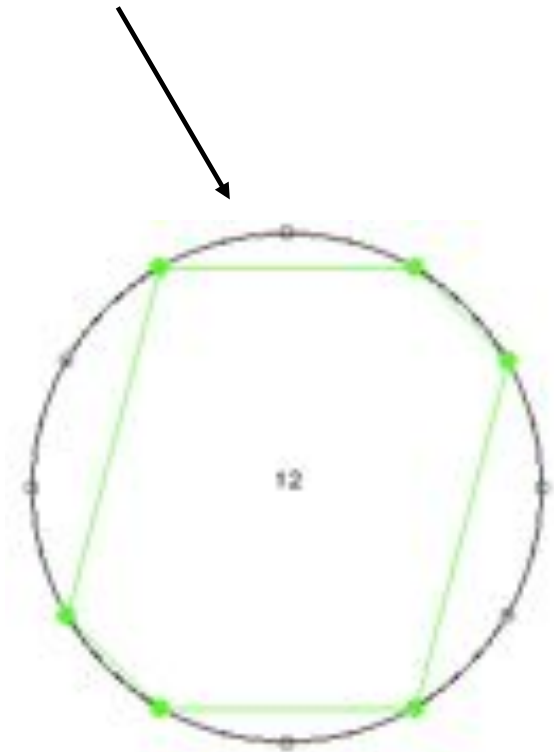
Exercice : retrouver les symétries dans une série (II)

Schoenberg: Serenade Op.24, Mouvement 5

$\{9, 10, 0, 3, 4, 6\}$ $\{5, 7, 8, 11, 1, 2\}$



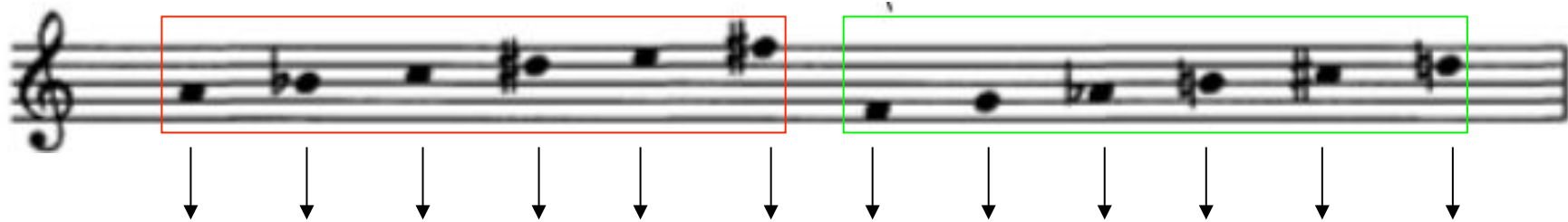
(3, 1, 2, 3, 1, 2)



(2, 1, 3, 2, 1, 3)

“Combinatorialité” et symétrie par transposition

Schoenberg: Serenade Op.24, Mouvement 5

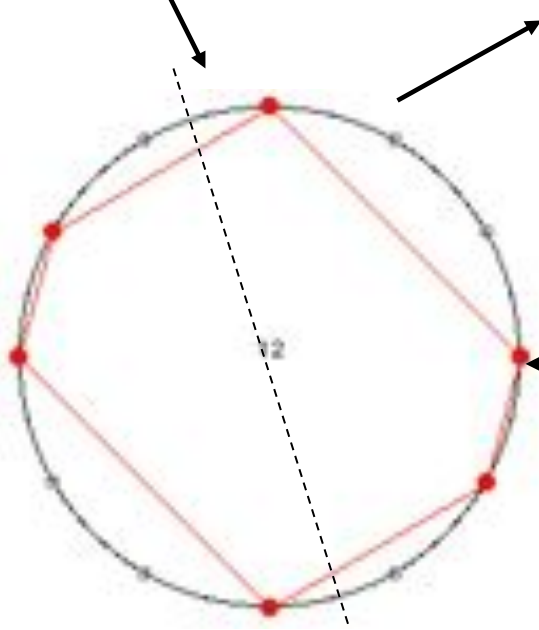


$$A = \{9, 10, 0, 3, 4, 6\} \quad \{5, 7, 8, 11, 1, 2\} = A'$$

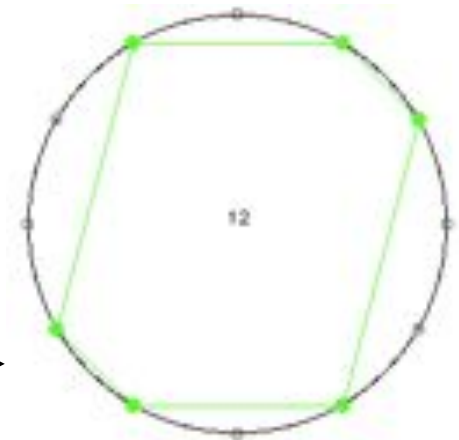
$$\begin{aligned} T_6\{9,10,0,3,4,6\} &= \\ &= \{6+9, 6+10, 6, 6+3, 6+4, 6+6\} = \\ &= \{3, 4, 6, 9, 10, 0\} \end{aligned}$$

$$T_6(A) = A$$

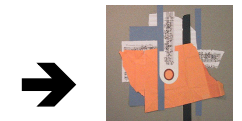
$$I_{11} = T_{11} I$$



(3, 1, 2, 3, 1, 2)



(2, 1, 3, 2, 1, 3)



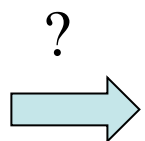
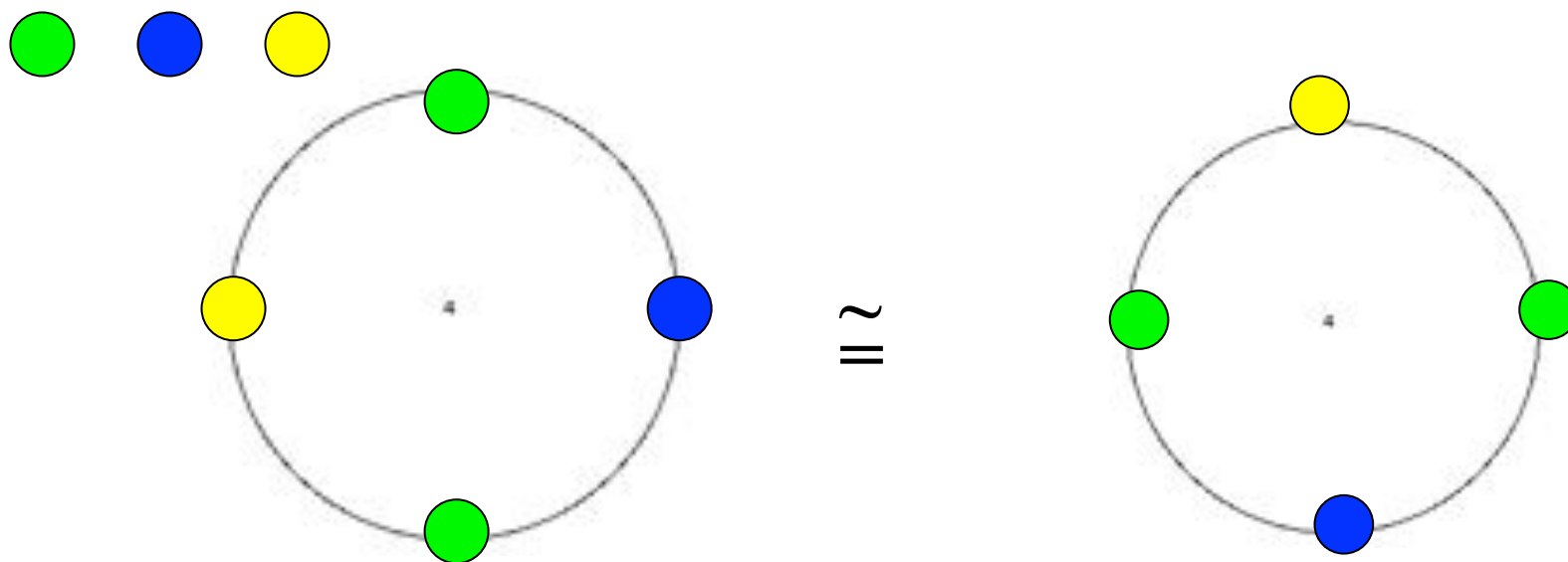
Enumeration des orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Trouver le nombre de configurations possibles

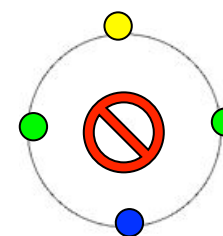
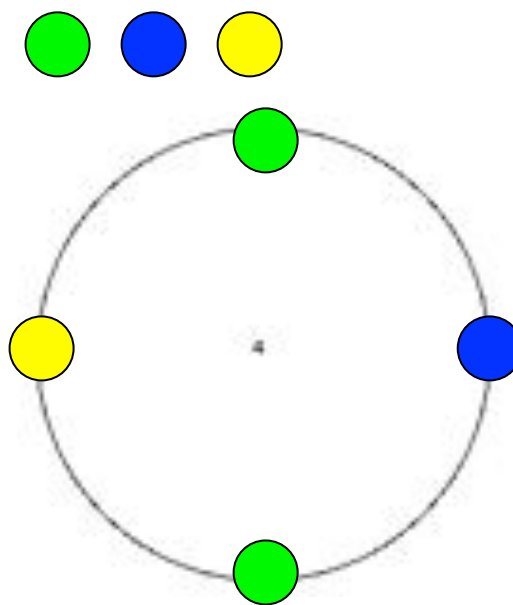
Enumération d'orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de $\mathbf{Z}/4\mathbf{Z}$

T_0 = identité

T_1 = rotation de 90°

T_2 = rotation de 180°

T_3 = rotation de 270°

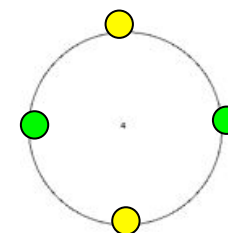
Configurations possibles = $3^4 = 81$

T_0 fixe toute configuration $\Rightarrow |X^{T_0}| = 81$

T_1 fixe toute configuration monochromes $\Rightarrow |X^{T_1}| = 3$

T_3 idem

T_2 fixe toute configuration «double-diamètre» $\Rightarrow |X^{T_2}| = 3^2 = 9$



➡ $n = 1/4 (81+3+3+9) = 24$

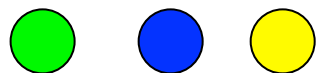
Enumération d'orbites par rapport à l'action d'un groupe



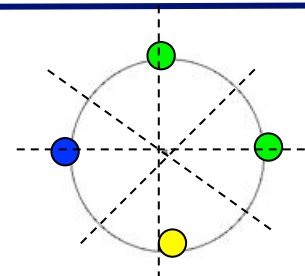
Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de Z_4



<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0\ 1\ 2\ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0\ 2)(1\ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0\ 3\ 2\ 1)$	1	$3^1 = 3$	4^1	t_4^1

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

$$n = 1/4 (81+3+3+9) = 24$$

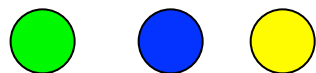
Enumération d'orbites par rapport à l'action d'un groupe



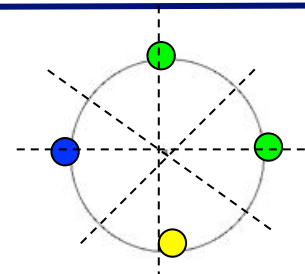
Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$



Action de D_4



<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0\ 1\ 2\ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0\ 2)(1\ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0\ 3\ 2\ 1)$	1	$3^1 = 3$	4^1	t_4^1
I	$0 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1, 2 \rightarrow 2$	$(0)(1\ 3)(2)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_1 I$	$0 \rightarrow 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 2$	$(0\ 1)(2\ 3)$	2	$3^2 = 9$	2^2	t_2^2
$T_2 I$	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 1, 3 \rightarrow 3$	$(0\ 2)(1)(3)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_3 I$	$0 \rightarrow 3 \rightarrow 0, 1 \rightarrow 2 \rightarrow 1$	$(0\ 3)(1\ 2)$	2	$3^2 = 9$	2^2	t_2^2

Julian Hook, « Why are there 29 Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory », MTO, 13(4), 2007

$$n = 1/8 (81+3+3+9+27+9+27+9) = 168/8=21$$

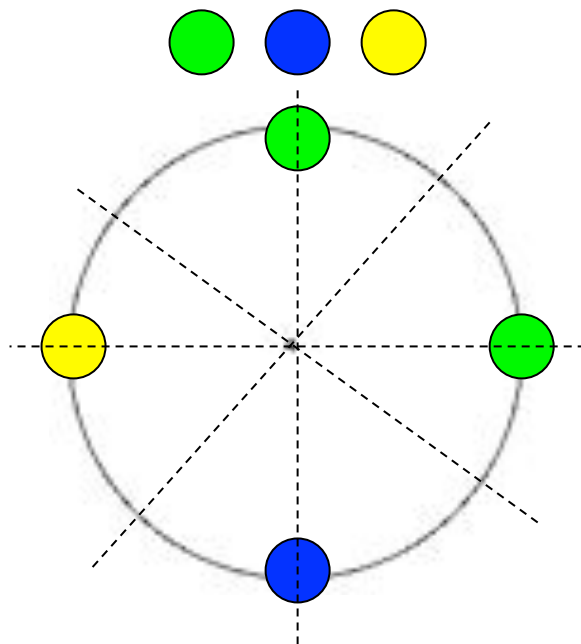
Enumération d'orbites par rapport à l'action d'un groupe



Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

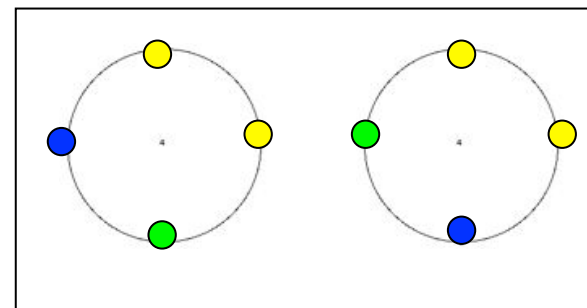
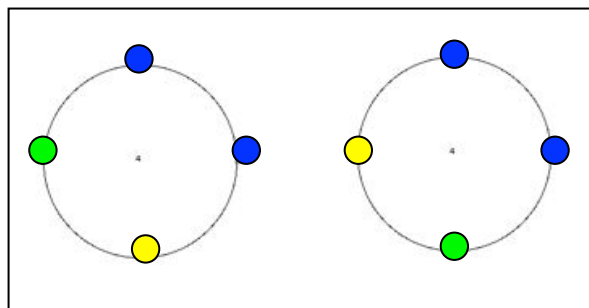
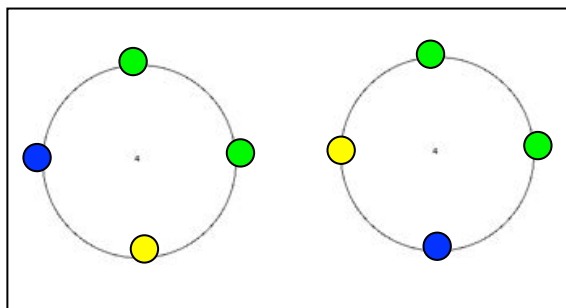
$X^g = \{x \in X : gx = x\}$



Action de D_4

- | | |
|-------------------------------|---------------------------|
| $T_0 = \text{id}$ | $T_0I = \text{inversion}$ |
| $T_1 = \text{rot } 90^\circ$ | $T_1I = \text{inv.}$ |
| $T_2 = \text{rot } 180^\circ$ | $T_2I = \text{inv.}$ |
| $T_3 = \text{rot } 270^\circ$ | $T_3I = \text{inv.}$ |

→ 21=24-3



Énumération d'accords par rapport à l'action du groupe diédrale



Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1

Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$$X^g = \{x \in X : gx = x\}$$

Action de Z_{12}

(Hook, MTO)



$$\# \text{ d'accords} = 1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352$$

Énumération d'accords par rapport à l'action du groupe diédrale



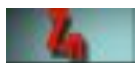
Transformation	Cycle representation	No. of cycles	Fixed configs.	Cycle type	Cycle index
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	f_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	f_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	f_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	f_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	f_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	f_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	f_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	f_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	f_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	f_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	f_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	f_{12}^1
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_1 I$	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	f_2^6
$T_2 I$	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_3 I$	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	f_2^6
$T_4 I$	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_5 I$	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	f_2^6
$T_6 I$	(0 6)(1 5)(2 4)(3)(7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_7 I$	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	f_2^6
$T_8 I$	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_9 I$	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	f_2^6
$T_{10} I$	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$f_1^2 f_2^5$
$T_{11} I$	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	f_2^6

Lemme de Burnside

$$n = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

$X^g = \{x \in X : gx = x\}$

Action de D_{12}
(Hook, MTO)



d'accords = $1/12[4096+2+4+8+16+2+64+2+16+8+4+2]=4224/12=352$



d'accords = $1/24[4224+1152] = 224$