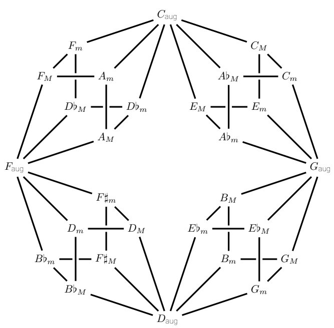
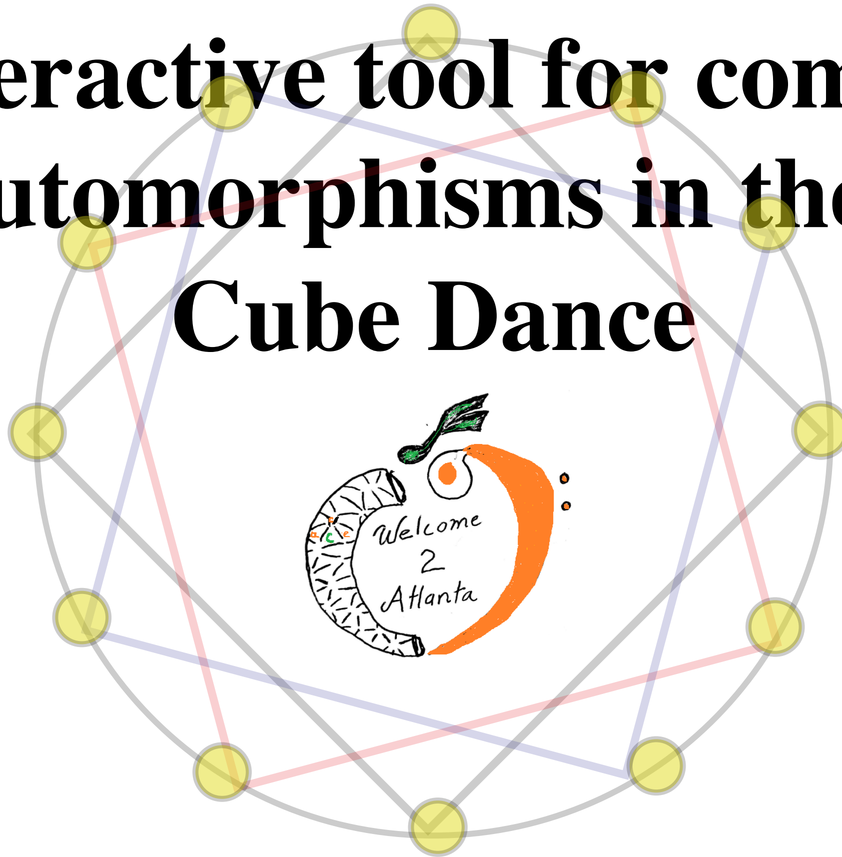
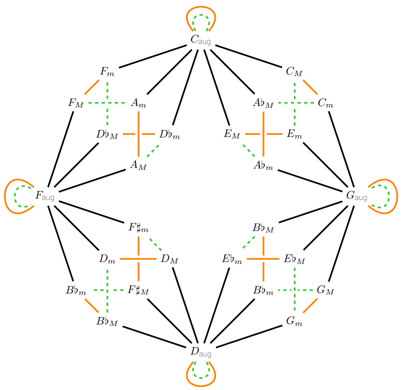
An interactive tool for composing (with) automorphisms in the colored Cube Dance



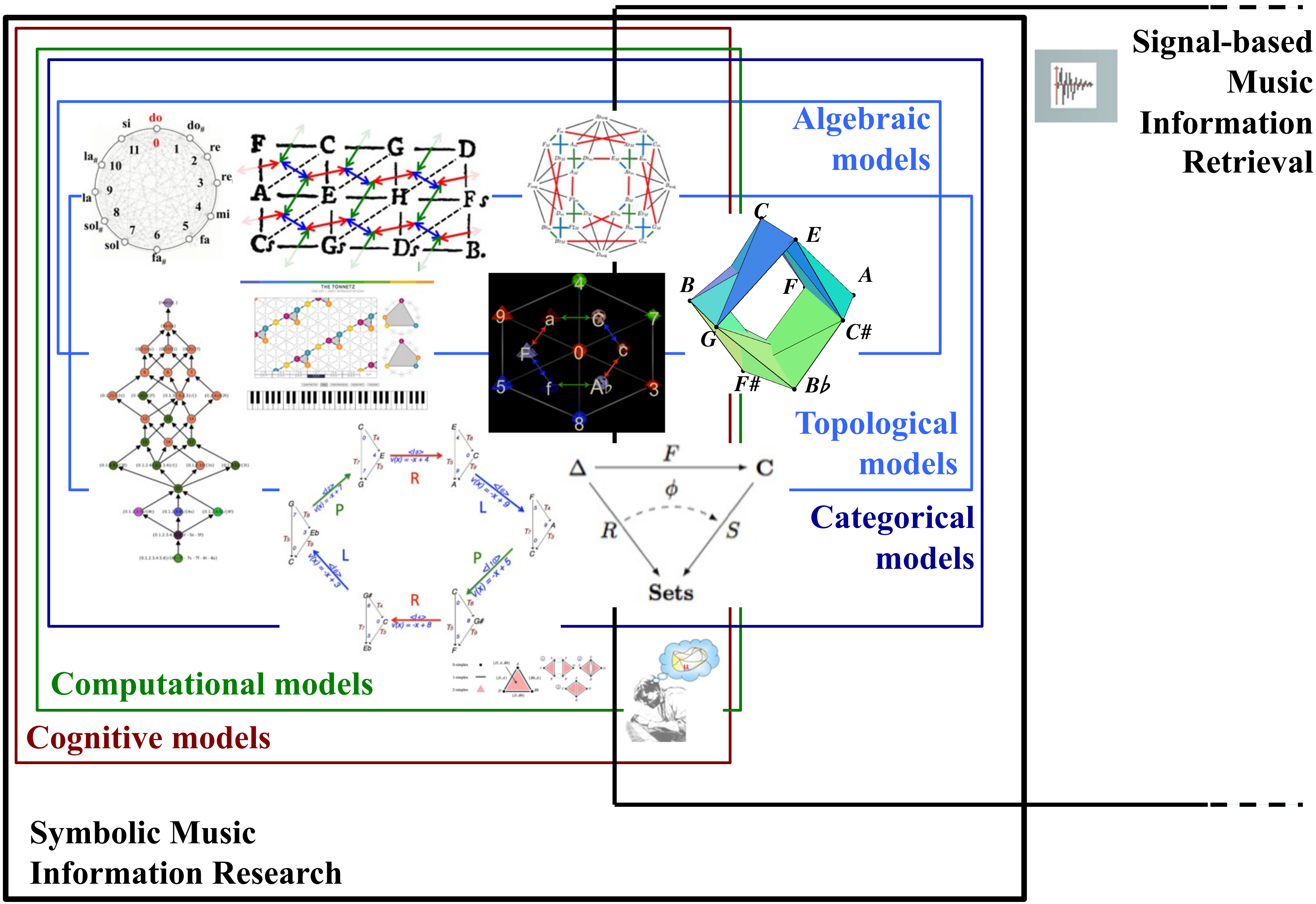




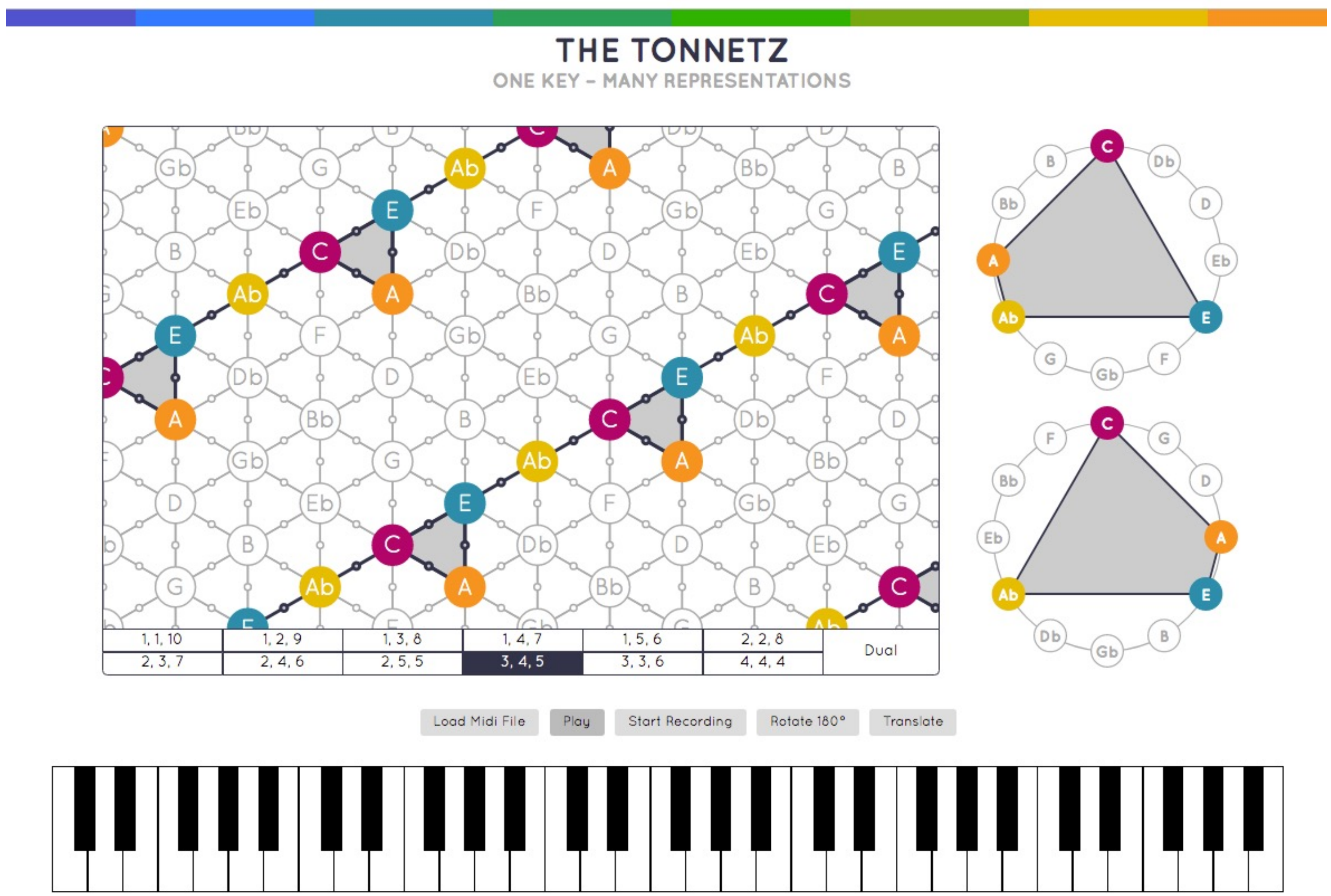
Alexandre Popoff, Corentin Guichaoua
and Moreno Andreatta

CNRS IRMA Institut de Recherche Mathématique Avancée
Université de Strasbourg
ircam Centre Pompidou
SORBONNE UNIVERSITÉ

The SMIR Project: advanced maths for the working musicologist



The *Tonnetz* web environment (© SMIR Project)



https://guichaoua.gitlab.io/web-hexachord/

A computational model of the Cube Dance (and its extensions)

Parsimonious graphs on triads for Douthett's and Steinbach's $P_{m,n}$ relations

➔ https://alexpof.github.io/interactive_mathmusic/Pmn_graphs/pmn_graphs.html

Select chords to display:

- ○ Major / Minor chords
- ◉ Major / Minor / Augmented chords
- ○ Major / Minor / Augmented / Sus4 chords

Select $P_{m,n}$ relations to display:

- ☑ $P_{1,0}$
- ☐ $P_{0,1}$
- ☐ $P_{1,1}$
- ☐ $P_{2,0}$
- ☐ $P_{0,2}$
- ☐ $P_{2,1}$
- ☐ $P_{1,2}$

Play Erase

Add chords to the progression by shift-clicking on the nodes.

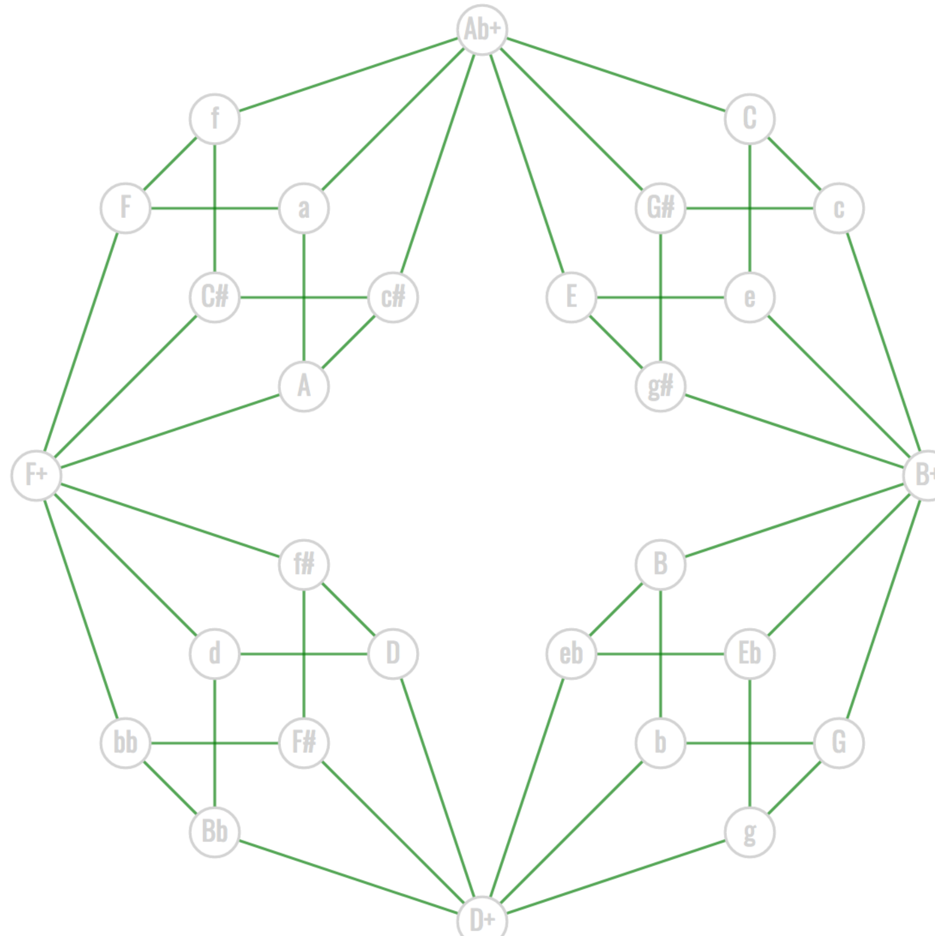
Two triads are said to be $P_{m,n}$-related if *m* pitch classes move by a semitone, while *n* pitch classes move by a whole tone, the rest of the pitch classes being identical.

Based on the original paper of Douthett and Steinbach : Douthett, Jack, and Peter Steinbach. 1998. "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition." Journal of Music Theory 42 (2): 241–263.

Visualization and code by Alexandre Popoff.
Best viewed with Chrome or Firefox. Compatibility with Internet Explorer and Microsoft Edge is not guaranteed.

Jack Douthett

Alexandre Popoff



- The $\mathcal{P}_{1,0}$ binary relation connects two chords if they differ by the movement of only one pitch class by one semitone.

[1]Douthett, J., Steinbach, P., Journal of Music Theory, 42(2), 1998, pp. 241–263. // Cohn, R. 'Audacious Euphony: Chromaticism and the Triad's Second Nature', Oxford University Press, 2012

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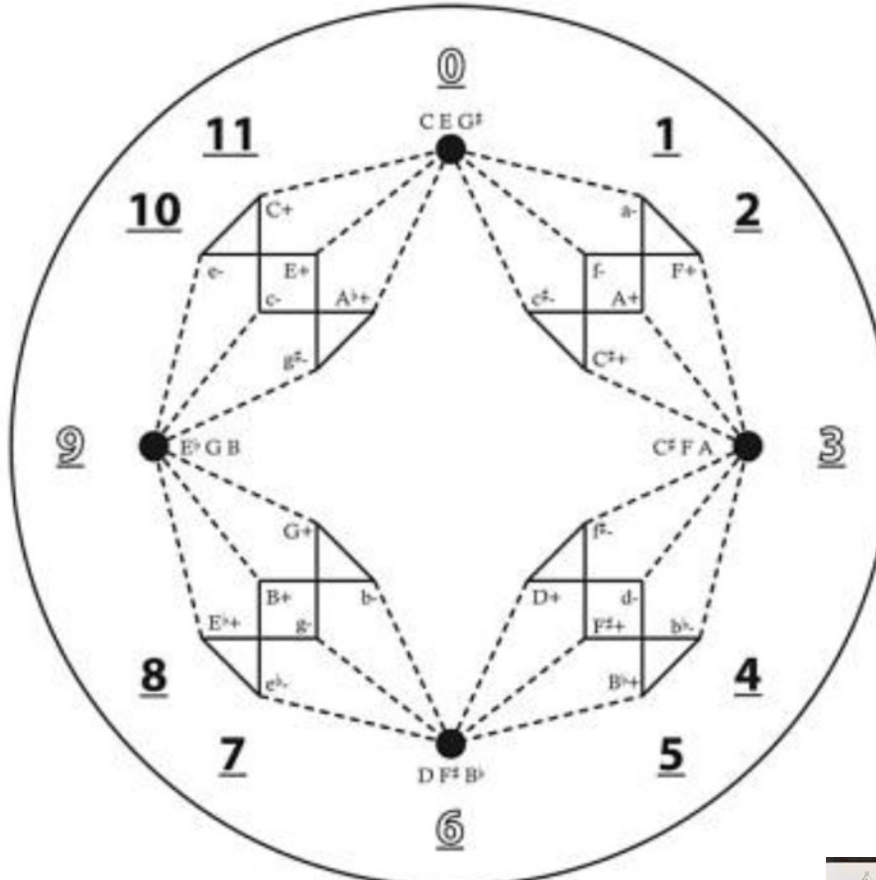

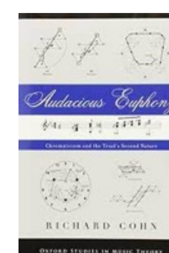

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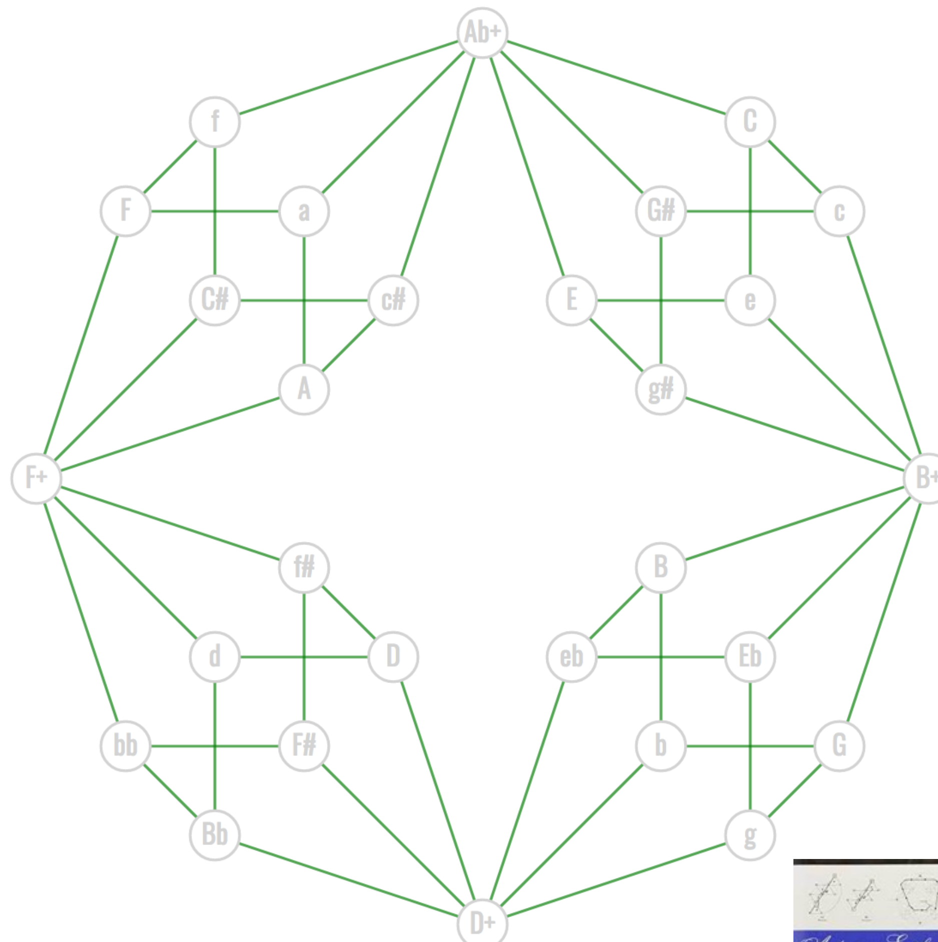

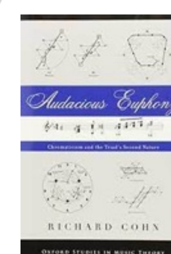

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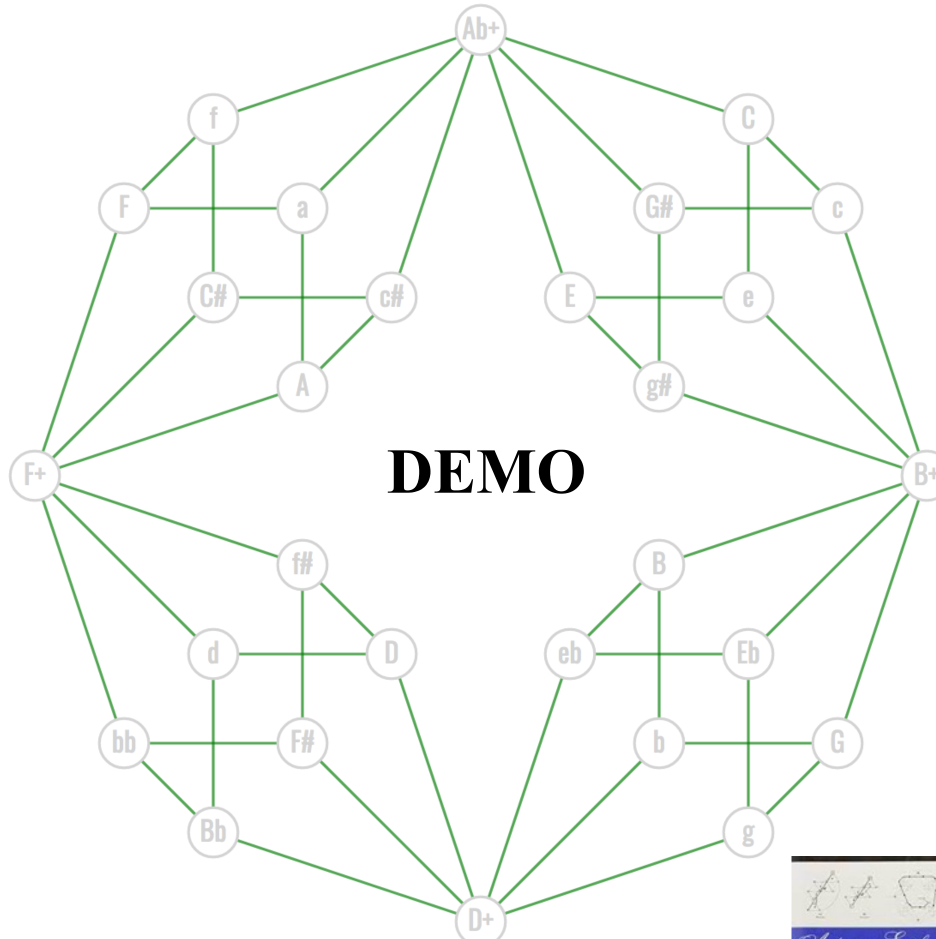
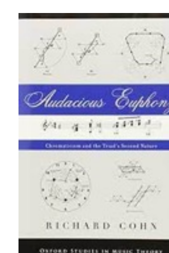


Jack Douthett

Alexandre Popoff

DEMO



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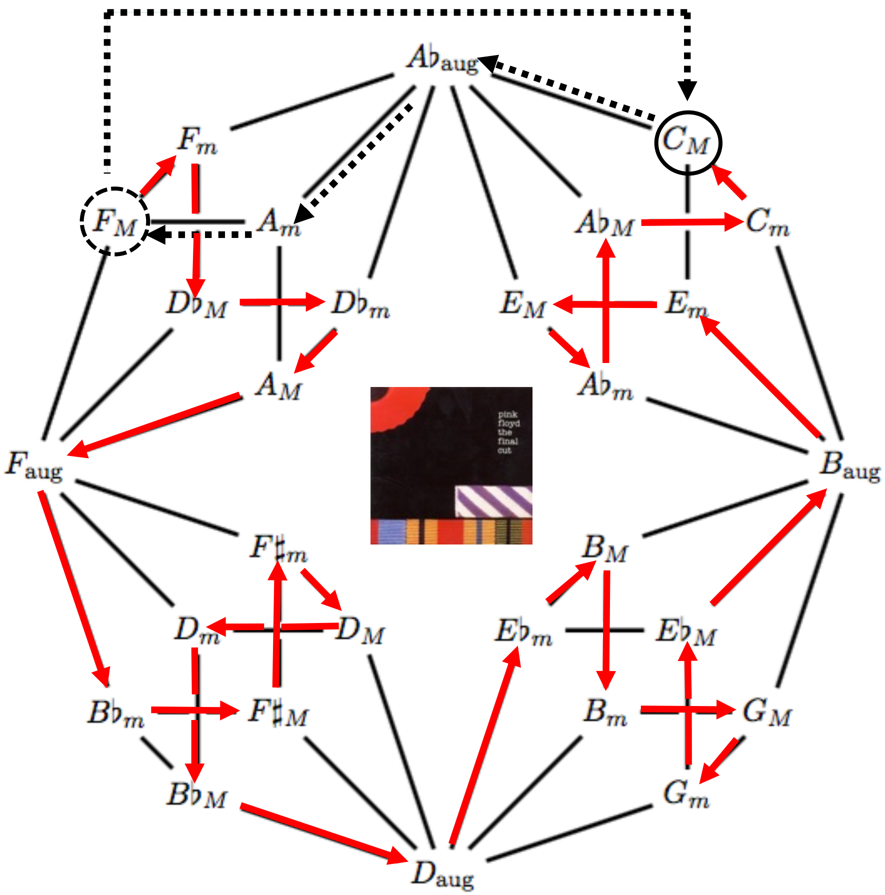
Composing with Hamiltonian Cycles in the 'classical' Cube Dance



The Gunner's dream (R. Waters, 1983 / M. Andreatta, 2018)

C C+
Floating down through the clouds
Am F
Memories come rushing up to meet me now.
Fm
In the space between the heavens
C# C#m
and in the corner of some foreign field
A F+ Bbm
I had a dream.
F# F#m D Dm
I had a dream.
Bb
Good-bye Max.
D+
Good-bye Ma.
Ebm B
After the service when you're walking slowly to the car
Bm G
And the silver in her hair shines in the cold November air

Gm
You hear the tolling bell
Eb
And touch the silk in your lapel
G+ Em E G#m
And as the tear drops rise to meet the comfort of the band
G# Cm
You take her frail hand
C
And hold on to the dream.

The three Hamiltonian Cycles (C_M = C, C_m = Cm, C_{aug} = C+)

C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->Bbm-->F#-->F#m-->D-->Dm-->Bb-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Em-->E-->G#m-->G#-->Cm-->C

C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->F#m-->F#-->Bbm-->Bb-->Dm-->D-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Em-->E-->G#m-->G#-->Cm-->C

C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->F#m-->D-->Dm-->Bb-->Bbm-->F#-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Cm-->G#-->G#m-->E-->Em-->C

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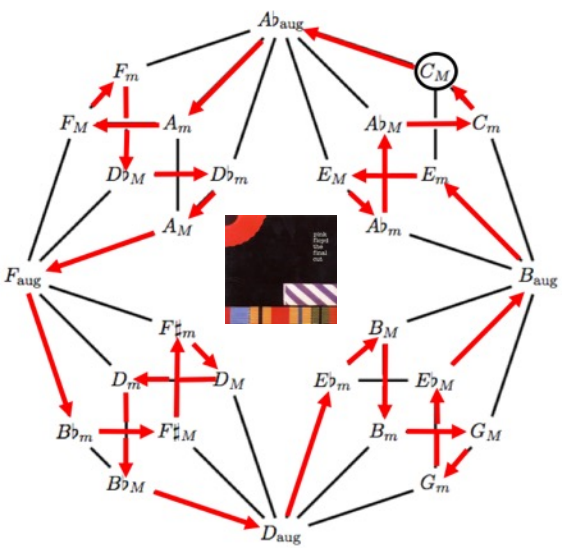

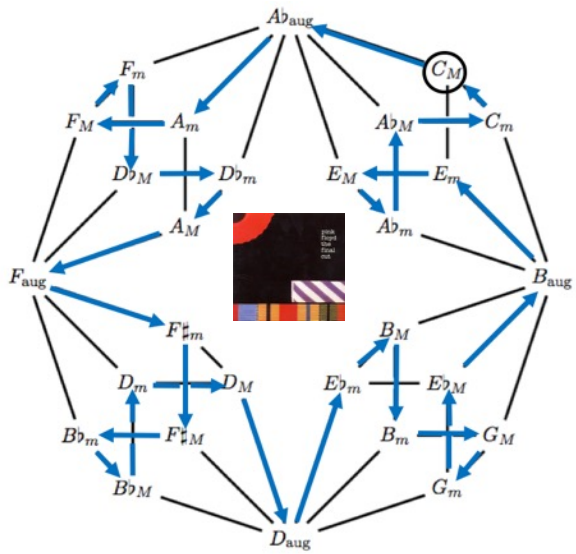

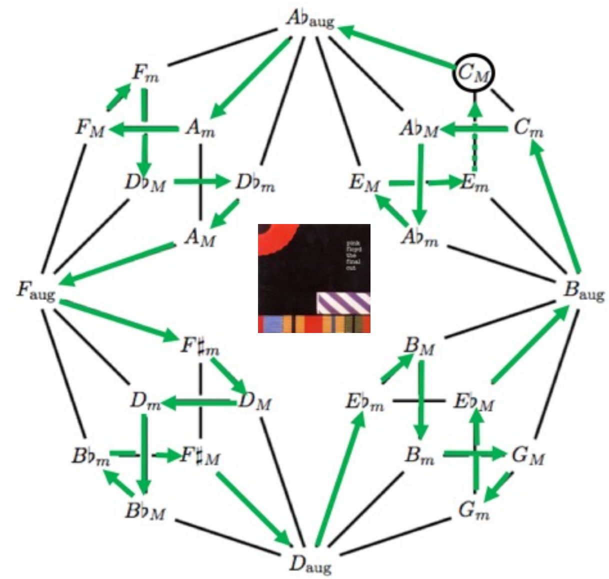

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C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->Bbm-->F#-->F#m-->D-->Dm-->Bb-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Em-->E-->G#m-->G#-->Cm-->C

C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->F#m-->F#-->Bbm-->Bb-->Dm-->D-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Em-->E-->G#m-->G#-->Cm-->C

C-->C+-->Am-->F-->Fm-->C#-->C#m-->A-->F+-->F#m-->D-->Dm-->Bb-->Bbm-->F#-->D+-->Ebm-->B-->Bm-->
-->G-->Gm-->Eb-->G+-->Cm-->G#-->G#m-->E-->Em-->C

HamilFloyd

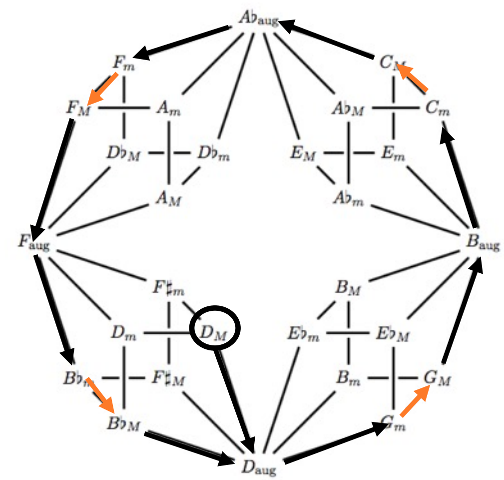
The Gunner's Hamiltionan Dream

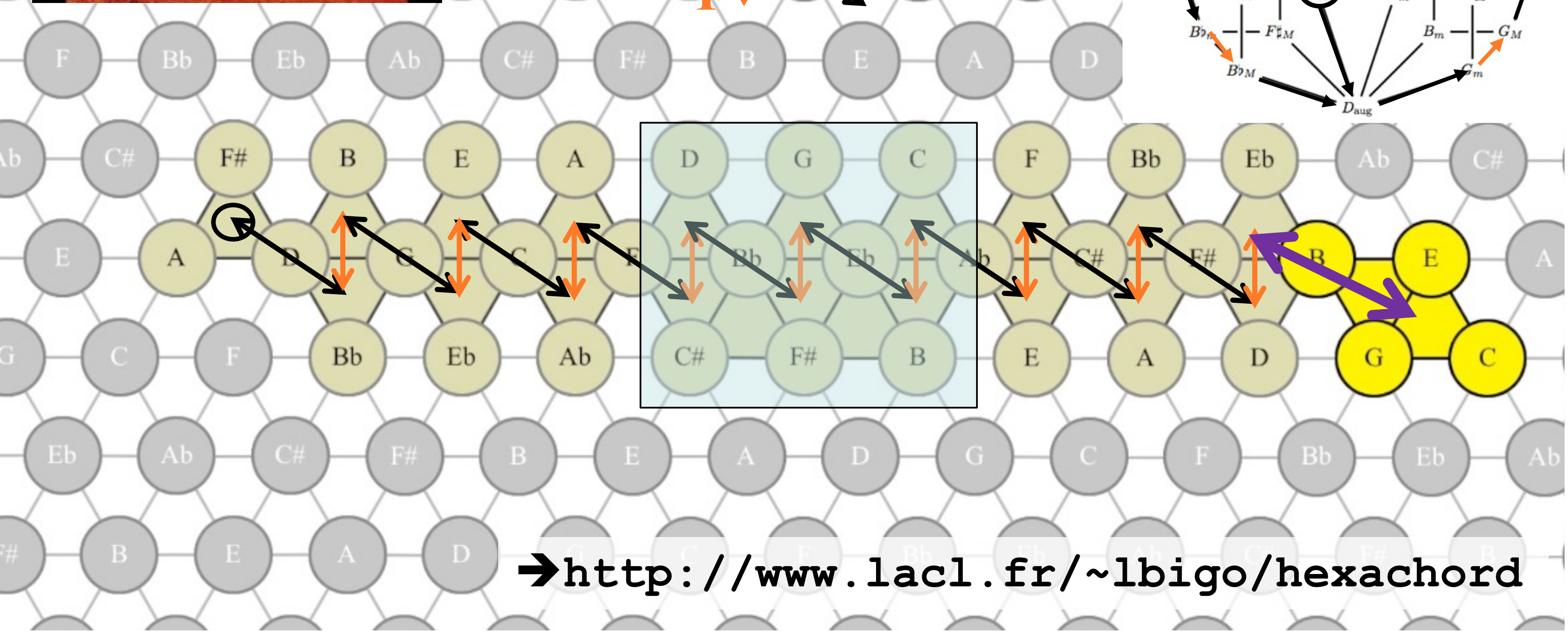
Moreno Andreatta
Gilles Baroin 2021

➔ **See you at the concert tribute to Jack Douthett (Thursday Evening)**

Tonnetz *versus* Cube Dance Analysis for Muse's *Take a bow*

"Take a bow" (*Black Holes and Revelations*, 2006)



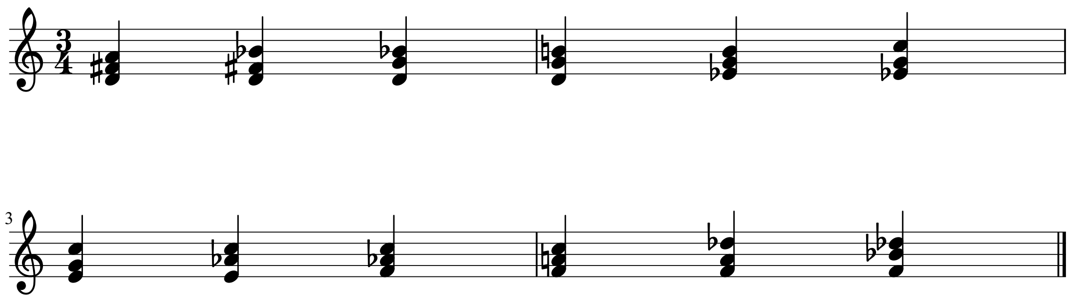


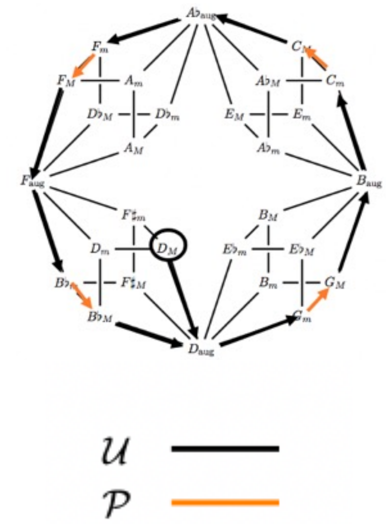
➔http://www.lacl.fr/~lbigo/hexachord

Muse - Take A Bow (Tonnetz harmonic analysis)

30,364 views Jan 20, 2016 Harmonic analysis of the song Take A Bow composed by Matthew Bellamy performed by Muse.

Two analytical approaches to Muse’s *Take a bow*





- First analytical approach: $\mathcal{U}$-$\mathcal{U}$-$\mathcal{P}$ cycle

$$\begin{tikzcd}
D_M \arrow[r, "\mathcal{U}"] & D_{\text{aug}} \arrow[r, "\mathcal{U}"] & G_m \arrow[r, "\mathcal{P}"] & G_M \arrow[r, "\mathcal{U}"] & B_{\text{aug}} \arrow[r, "\mathcal{U}"] & C_m \arrow[dlllll, "\mathcal{P}"] \\
C_M \arrow[r, "\mathcal{U}"'] & A\flat_{\text{aug}} \arrow[r, "\mathcal{U}"'] & F_m \arrow[r, "\mathcal{P}"'] & F_M \arrow[r, "\mathcal{U}"'] & F_{\text{aug}} \arrow[r, "\mathcal{U}"'] & B\flat_m
\end{tikzcd}$$

- Second analytical approach: transpositions by fourth of the first cell

$$\begin{tikzcd}
D_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & G_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & C_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & F_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & \\
& D_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & & B_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & & A\flat_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & & F_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
G_m & & C_m & & F_m & & B\flat_m &
\end{tikzcd}$$

A category-based approach of transformational analysis

$$
\begin{tikzcd}
C \arrow[r, "T_3"'] \arrow[d, leftrightarrow, "I_2"'] & E\flat \arrow[d, "T_7M_7"'] \\
D \arrow[r, "T_2"'] & E
\end{tikzcd}
\quad
\begin{tikzcd}
E\flat \arrow[r, "T_3"'] \arrow[d, leftrightarrow, "I_2"'] & F\sharp \arrow[d, "T_7M_7"'] \\
B \arrow[r, "T_2"'] & C\sharp
\end{tikzcd}
\quad
\begin{tikzcd}
F\sharp \arrow[r, "T_3"'] \arrow[d, leftrightarrow, "I_2"'] & A \arrow[d, "T_7M_7"'] \\
G\sharp \arrow[r, "T_2"'] & B\flat
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "T_3"'] \arrow[d, leftrightarrow, "I_2"'] & C \arrow[d, "T_7M_7"'] \\
F \arrow[r, "T_2"'] & G
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
\square \arrow[rr, dash, "F"] \arrow[dr, dash, "R"'] & \phi & \square \arrow[dl, dash, "S"] \\
& \square &
\end{tikzcd}
$$

Definition 1 *Let* $\mathbf{C}$ *be a category, and* S *a functor from* $\mathbf{C}$ *to the category* $\mathbf{Sets}$. *Let* Δ *be a small category and* R *a functor from* Δ *to* $\mathbf{Sets}$. *A PK-net of form* R *and of support* S *is a 4-tuple* (R, S, F, ϕ), *in which*

- F *is a functor from* Δ *to* $\mathbf{C}$,
- *and* ϕ *is a natural transformation from* R *to* SF.

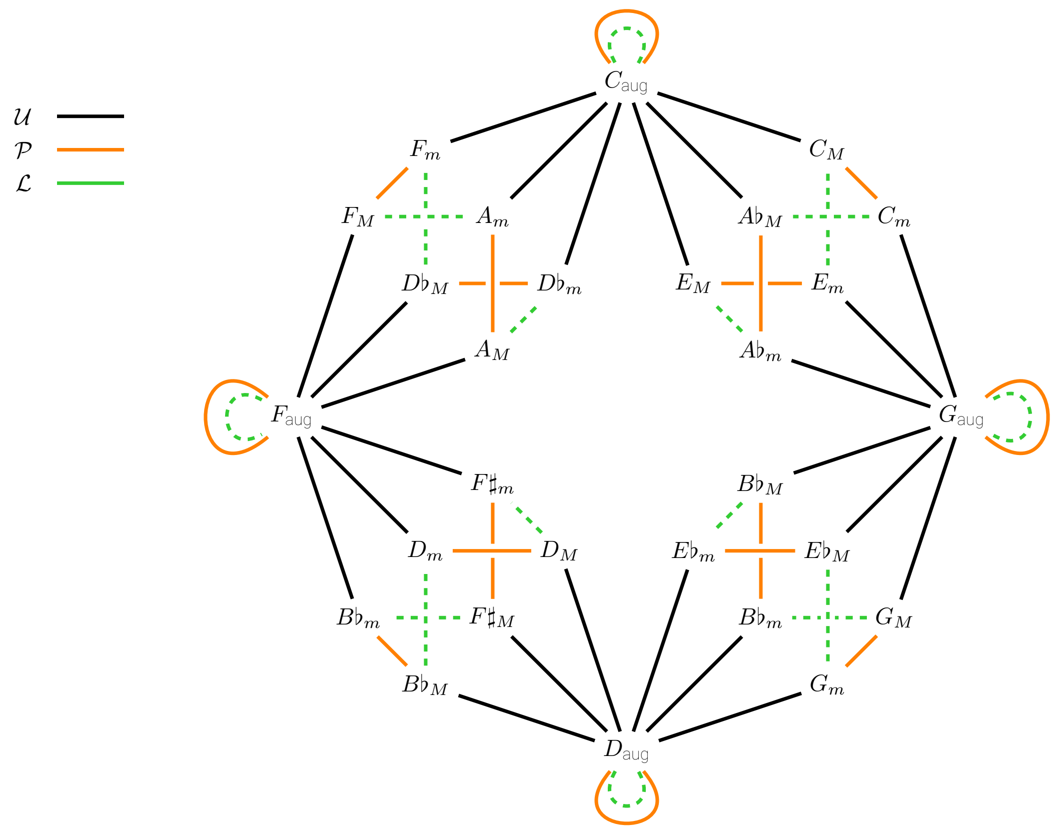
The definition of a PK-net is summed up by the following diagram:

$$
\begin{tikzcd}
\Delta \arrow[rr, "F"] \arrow[dr, "R"'] & \phi & \mathbf{C} \arrow[dl, "S"] \\
& \mathbf{Sets} &
\end{tikzcd}
$$

Popoff A., M. Andreatta, A. Ehresmann, « A Categorical Generalization of Klumpenhouwer Networks », MCM 2015, Queen Mary University, Springer, p. 303-314

From the Cube Dance to the colored Cube Dance

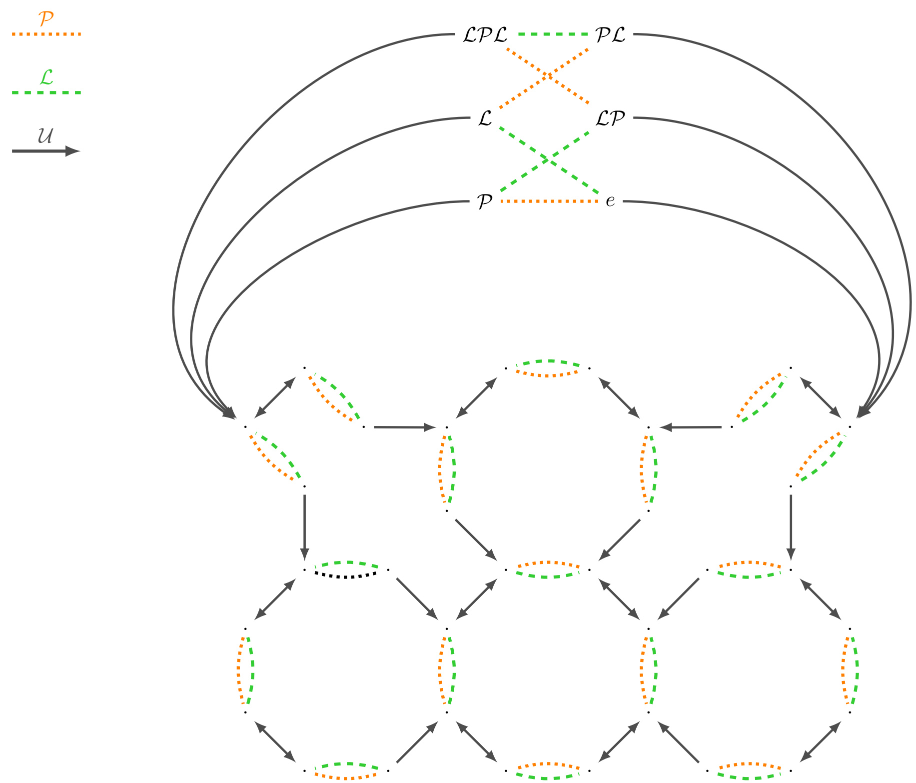
Refining the $\mathcal{P}_{1,0}$ relation: the ‘Colored Cube Dance’



- We consider three sub-relations of $\mathcal{P}_{1,0}$ named $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$
- The relations $\mathcal{P}$ and $\mathcal{L}$ coincide on major and minor chords with the usual neo-Riemannian operations.

A ‘weak’ algebraic structure

The $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ monoid



The monoid $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ generated by $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{L}$ contains 40 elements and has for presentation

$$\begin{aligned} M_{\mathcal{U},\mathcal{P},\mathcal{L}} = \langle \mathcal{U}, \mathcal{P}, \mathcal{L} \mid \mathcal{P}^2 = \mathcal{L}^2 = e,\ \ \mathcal{LPL} = \mathcal{PLP},\ \ \mathcal{U}^3 = \mathcal{U}, \\ \mathcal{UP} = \mathcal{UL},\ \ \mathcal{PU} = \mathcal{LU},\ \ \mathcal{U}^2\mathcal{PU}^2 = \mathcal{PU}^2\mathcal{PU}^2\mathcal{P}, \\ (\mathcal{UP})^2\mathcal{U}^2 = \mathcal{P}(\mathcal{UP})^2\mathcal{U}^2\mathcal{P},\ \ \mathcal{U}^2(\mathcal{PU})^2 = \mathcal{PU}^2(\mathcal{PU})^2\mathcal{P} \rangle \end{aligned}$$

The algebraic set-up: relational PK-nets

Automorphisms of the $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ action

$$
\begin{tikzcd}
D_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & & G_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & & C_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & & & F_M \arrow[dd, "\mathcal{U}^2"'] \arrow[dr, "\mathcal{U}"] & \\
& D_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & {} & & B_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & {} & & A\flat_{\text{aug}} \arrow[dl, "\mathcal{U}"] \arrow[r] & {} & & F_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
G_m & & & C_m & & & F_m & & & B\flat_m &
\end{tikzcd}
$$

- Transposition by fourth is a special example of an automorphism of the action of $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ on the set X of major, minor, and augmented triads.

Definition

The automorphism group A of the action of $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ on X is the group of pairs (N, ν) where

- $N\colon M_{\mathcal{U},\mathcal{P},\mathcal{L}} \to M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ is an automorphism, and
- ν is a bijection on X, such that
- we have $p\mathcal{R}q \implies \nu(p)N(\mathcal{R})\nu(q)$ for all $\mathcal{R} \in M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ and $(p, q) \in X^2$.

Composition is done term-wise.[2]

- Problem: can we determine the structure of A ?

The algebraic set-up: relational PK-nets

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$$
\begin{tikzcd}
D_M \arrow[dr, "\mathcal{U}"] \arrow[dd, "\mathcal{U}^2"'] & \\
& D_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
G_m &
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
G_M \arrow[dr, "\mathcal{U}"] \arrow[dd, "\mathcal{U}^2"'] & \\
& B_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
C_m &
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
C_M \arrow[dr, "\mathcal{U}"] \arrow[dd, "\mathcal{U}^2"'] & \\
& A\flat_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
F_m &
\end{tikzcd}
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\begin{tikzcd}
F_M \arrow[dr, "\mathcal{U}"] \arrow[dd, "\mathcal{U}^2"'] & \\
& F_{\text{aug}} \arrow[dl, "\mathcal{U}"] \\
B\flat_m &
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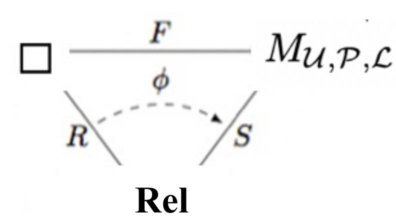


[2] For the more general definition of the automorphism group of a functor $S\colon M \to \mathbf{Rel}$, see Popoff, A., Andreatta, M., Ehresmann, A. 'Relational poly-Klumpenhouwer networks for transformational and voice-leading analysis.' J. Math. Music, 12(1), 2018, pp. 35–55

The algebraic set-up: relational PK-nets

Automorphisms of the $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ action

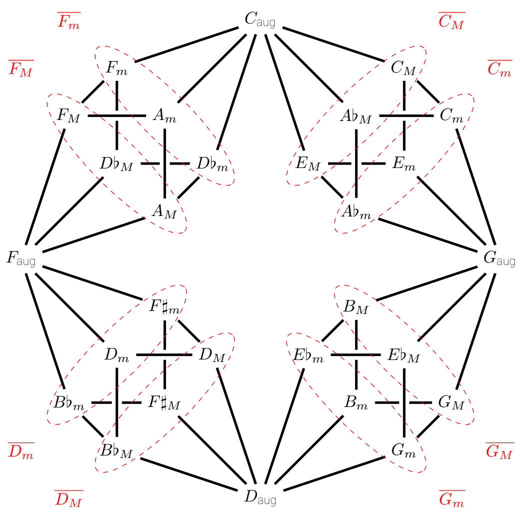
Theorem

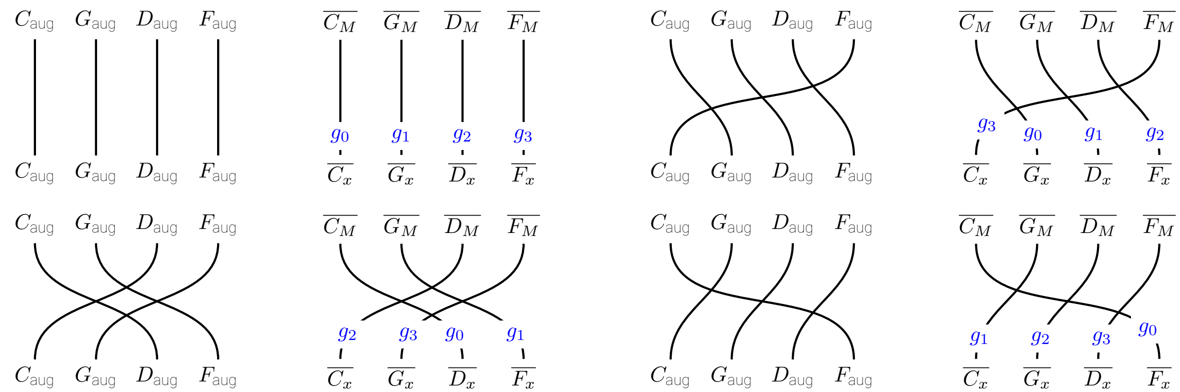
The automorphism group A of the action of $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ on X is a group of order 7776 isomorphic to $({\mathbb{Z}_3}^4 \rtimes D_8) \rtimes (D_6 \times \mathbb{Z}_2)$.

- The group $D_6 \times \mathbb{Z}_2$ corresponds to the automorphisms N of the monoid $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ itself.
  - D_6 corresponds to the automorphisms of the subgroup generated by $\mathcal{P}$ and $\mathcal{L}$.
  - $\mathbb{Z}_2$ corresponds to the image of $\mathcal{U}$ which can be either $\mathcal{U}$ or $\mathcal{LUL}$.
- The complete structure of A is determined by the careful enumeration of the possible bijections $\nu\colon X \to X$.

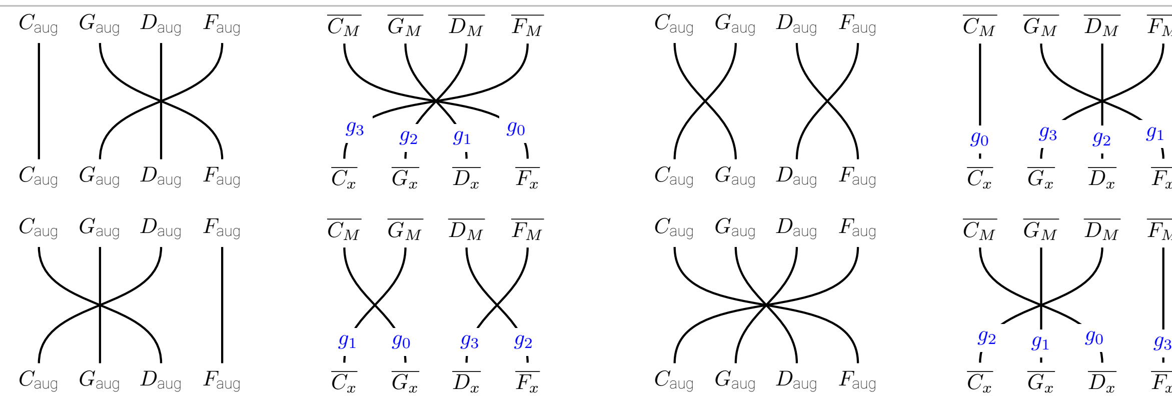
Graphical representation of the possible automorphisms

Automorphisms of the $M_{\mathcal{U},\mathcal{P},\mathcal{L}}$ action





$x = M$ if $N(\mathcal{U}) = \mathcal{U}$, $x = m$ if $N(\mathcal{U}) = \mathcal{LUL}$



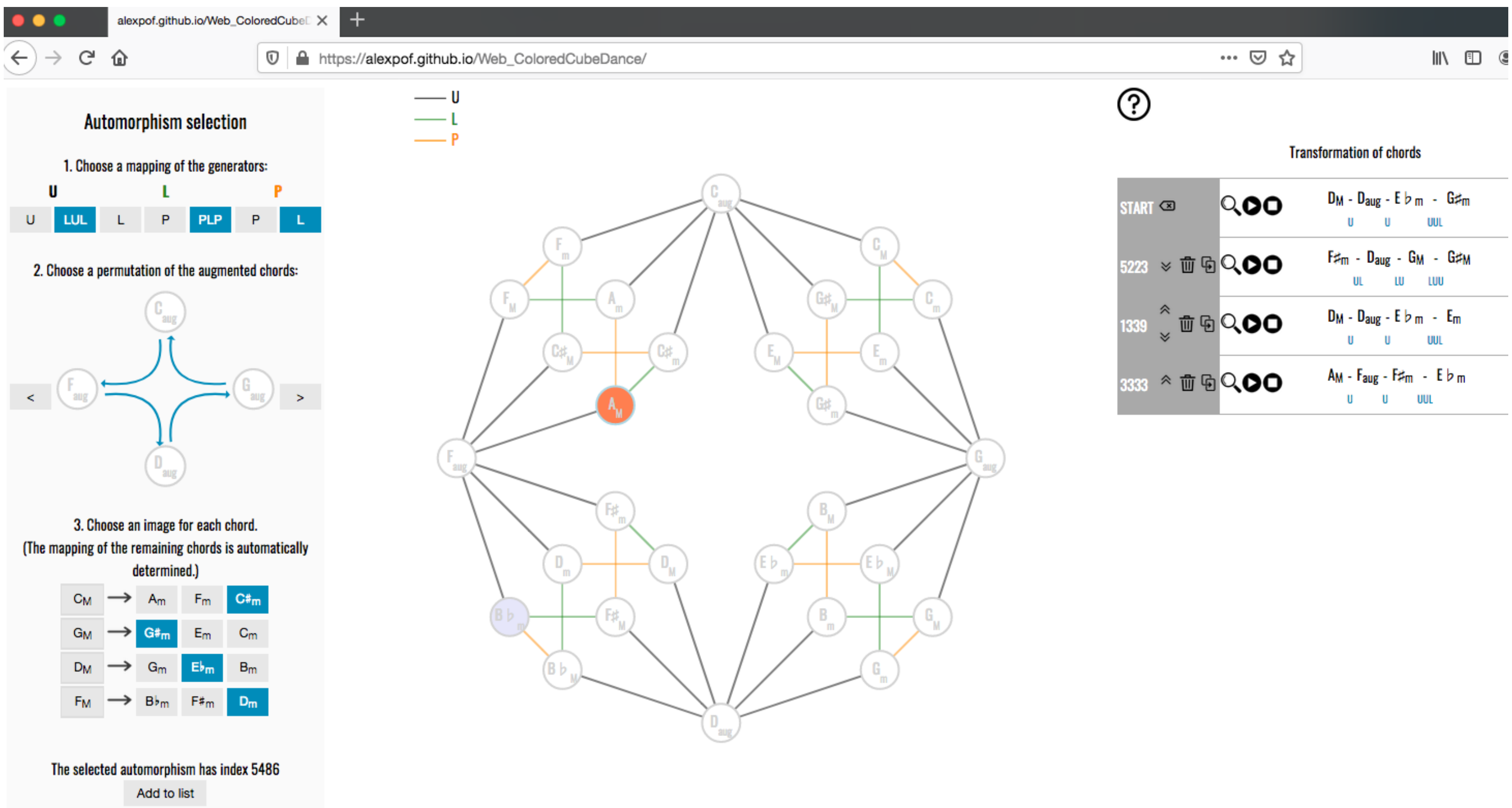
$x = m$ if $N(\mathcal{U}) = \mathcal{U}$, $x = M$ if $N(\mathcal{U}) = \mathcal{LUL}$

The g_i are elements of the subgroup generated by $\mathcal{PL}$ (isomorphic to $\mathbb{Z}_3$).

(see Table 1)

Composing (with) automorphisms

- The group $A \cong (\mathbb{Z}_3{}^4 \rtimes D_8) \rtimes (D_6 \times \mathbb{Z}_2)$ is too complex to be easily manipulated by hand.
- We developed an interactive interface intended for mathemusicians and composers for transforming chord progressions using elements of A.
- HTML/SVG (graphical elements) and Javascript (code) allows one to develop complex interfaces for outreach activities. No installation needed !



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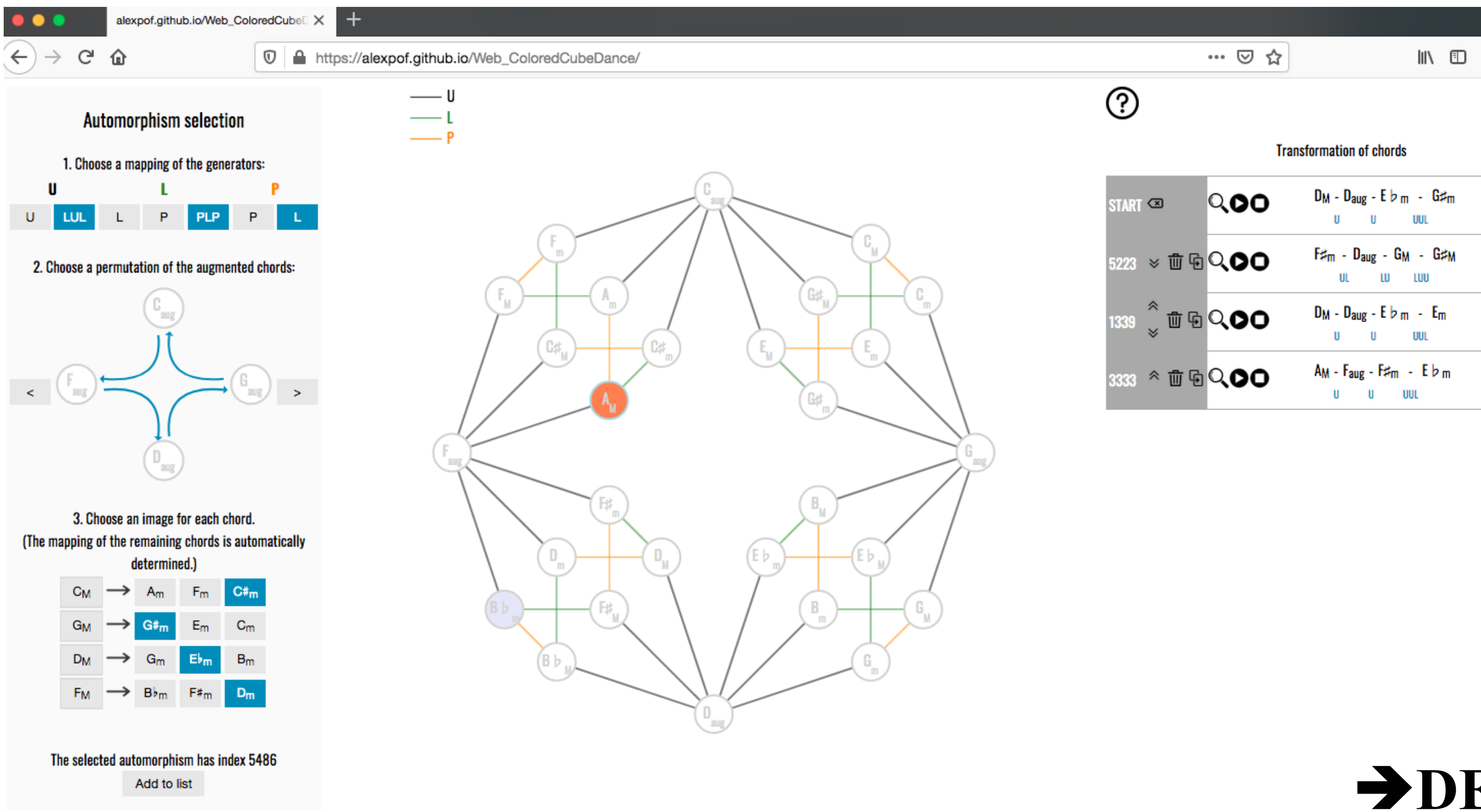

→DEMO

Thank you for your attention!