Tiling canons as a key to approaching open mathematical conjectures?

Moreno Andreatta

This chapter provides a first introduction to the formalization and construction of rhythmic tiling canons and their connections to interesting mathematical problems. We briefly provide a historical account of the emergence of tiling canon constructions, from Olivier Messiaen's model of non-invertible rhythmic canons to Dan Vuza's theory of supplementary sets and regular complementary unending canons. Tiling canons are prototypical examples of "mathemusical" research problems: the problem of constructing them, which was originally musical, when set in an appropriate mathematical framework not only gave rise, eventually, to new mathematical results, but also paved the way to new music-theoretic, analytical, and compositional constructions that would have been difficult to conceive without the process of mathematization and modeling. We discuss some computational aspects of rhythmic canon constructions, in particular with respect to the OpenMusic visual programming language, and then mention some compositional applications of the computer-aided model of tiling canon constructions by composers such as Fabien Lévy, Georges Bloch, Mauro Lanza, Daniele Ghisi, and Tom Johnson.

1. Introduction: retracing the double history of rhythmic tiling canons constructions

The history of tiling canons is particularly interesting because it shows how a truly musical problem may intersect with concepts and ideas belonging to the history of mathematics, from number theory and the geometry of tiling to operator theory in functional analysis.[a]

[a]The article summarizes some of the ideas discussed in more detail in [6]. The reader interested in the history of such a 'mathemusical' problem may find more information there as well as in the other contributions of the special issue of *Perspectives of New*

There is probably no need to explain in depth the relevance of canons to music since it is one of the few musical concepts that have been used extensively, well beyond the boundaries of the Western classical music tradition. One may naturally think of the complex polyphonic structures of the *Ars Nova* [32] and the way in which this model influenced contemporary music theorists and composers, from Bernhard Ziehn's *Canonic Studies* [38] to Olivier Messiaen's *Traité de rythme, de couleur et d'ornithologie* [25]. The French "rhythmitician," as Messiaen used to refer to himself, made undoubtedly one of the most significant efforts to study canons by focusing on the underlying rhythmic structure instead of the pitch content.

By definition, a *rhythmic canon* is a polyphonic setting of the same rhythm translated in time. It is defined by two rhythmic patterns: the inner one (R), a period rhythm that is the ground voice, and the outer one (S) defined by the timing of the entries of each voice. A *tiling rhythmic canon* is a rhythmic canon where there is no superposition between the voices of the canons meaning that at each pulse (for example, at each quarter note) one and only one voice attacks a note.

A brief analysis of Messiaen's compositional practice shows that the starting point is a genuine compositional problem related to some apparently very different theoretical constructions, such as the modal theory of the Rumanian composer Anatol Vieru (1926-1998)—in particular the concept of "composition" between modal structures [34]—and some serial techniques by the French conductor and composer Pierre Boulez, such as the concept of chord multiplication. The second of the seven volumes of Messiaen's *Traité de rythme, de couleur et d'ornithologie* entitled "Pedals and rhythmic canons" is in fact entirely devoted to the study of rhythmic structures including augmentation and diminution, irrational values and other rhythm-based canonic techniques. Apart from being the first comprehensive attempt at defining the form of musical canons by focusing exclusively on rhythmic organization, this treatise establishes a connection between rhythmic canons and Messiaen's favorite technique of "non-invertible" (or non-retrogradable) rhythms. These are defined as possessing "two groups of durations, one the retrograde of the other, surrounding [*encadrant*] a central free value which is common to the two groups" [25, p. 7]).

As Messiaen rightly observes, there is a formal equivalence between the

Music devoted to Tiling Problems in Music [31]. Mathematicians interested to the deep connections between open conjectures and musical tiling problems will find a collection of research articles in the special issue of the Journal of Mathematics and Music (Andreatta & Agon 2007).

non-invertibility and the palindromic character of a rhythm.[b] He explicitly uses the property of non-invertibility of rhythms, together with the operation of changing a minimal unit of a rhythmic pattern, to construct special families of rhythmic canons whose formal structure directly calls to mind the tiling concept in geometry. As mentioned, a rhythmic canon is the repetition, with a temporal translation, of a rhythmic structure (or its possible transformations). The base "inner" rhythmic pattern—which is called *pédale rythmique* in Messiaen's terminology—is repeated and translated in time mostly in a regular way.[c]

A particular type of rhythmic canon, recurrent in Messiaen's compositions, is obtained by considering as the inner rhythm a concatenation of non-invertible rhythms, like in the case of the part entitled "Amen des anges, des saints, du chant des oiseaux" of the piece *Visions de l'Amen* (1943) for two pianos, or the piece *Harawi* (1945) for soprano and piano. The fundamental aspect of this compositional process is the tension between non-invertible rhythms and the regular entries of voices, which is responsible for the global perceptual result of a mixture of chaotic behavior and organized structure, hence the expression "organized chaos" [25, 0. 46] used by the composer to describe it. Note that Messiaen never refers explicitly to the role of geometric and tiling processes in music. Nevertheless, this geometrical concept adequately captures the underlying compositional idea, although with a certain degree of "divergence" between the actual compositional result and the formal mathematical model.[d] It is clear that Messiaen aims at using a special family of concatenations of non-invertible rhythms in order to organize the global musical form as a canon in such a way as to make the onsets of the different rhythmic voices (potentially) never intersect. Nevertheless, this is not what finally happens in the actual compositions, since mutual intersections between voices frequently occur—particularly in the second half of the canon (see Figure 1).

Despite Messiaen's faith in palindromic structures as applied to the canonic process, it is easy to show that there is generally no connec-

[b]We do not enter here into the discussion of another formal equivalence, which is postulated by the composer, and which is based on the analogy between non-invertible rhythms and modes with limited transpositions. See [6] for a critical perspective on this misleading analogy. For an interesting analysis of Messiaen's conscious compositional use of mathematical concepts derived from symmetry and groups, see [29].

[c]As we will see, the regular entries of the voices of most of the canons in the literature is a property which no longer holds in Vuza Canons.

[d]For an epistemological discussion on the possible kinds of "divergence" between mathematical models and compositional processes, see [8].

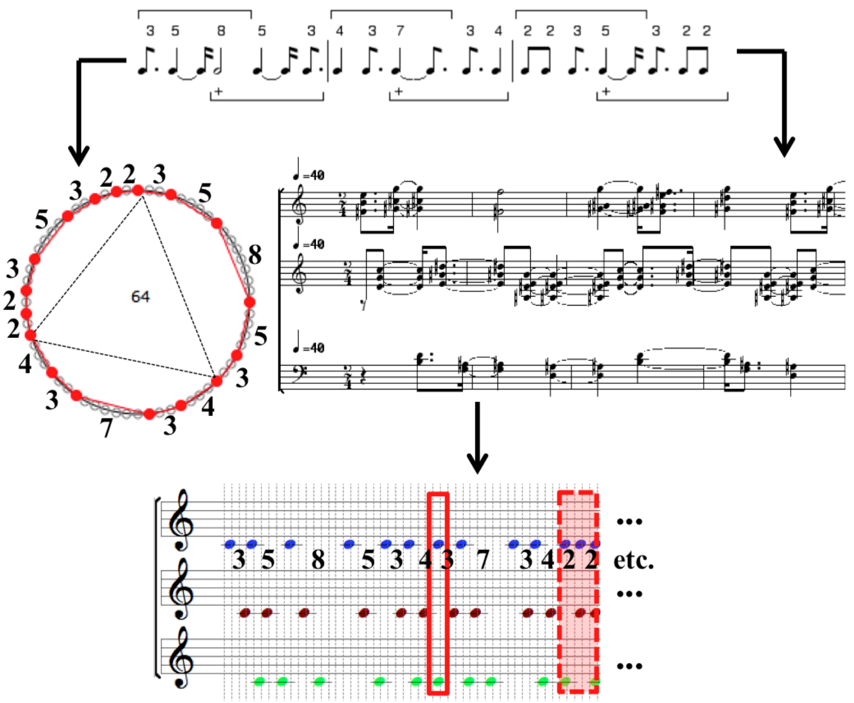

Fig. 1. The use of palindromic structures as the inner rhythm underlying the canonic process in *Harawi* (1945), together with the grid representation showing regular entries of voices and the increasing numbers of intersections between the voices of the triple non-invertible canon.

tion between non-invertible rhythms and tiling processes. There exist non-invertible rhythms that may eventually be taken as the "inner rhythm" of a rhythmic canon rigorously realizing the tiling of the time axis, as in most of the examples shown in the next section.[e] Nevertheless, Messiaen was, of course, unaware of the existence of concepts such as "group factorizations" or "direct sums," which are used today to elegantly describe the construction of tiling canons. This is one of Dan Tudor Vuza's major contributions to the field which take as a point of departure Anatol Vieru's concept of composition of modal structures and re-interprets Vieru's composition law in the time domain [35, 36].

[e]As an example, see Figure 3 where one only need to exchange the role of inner/outer rhythms (via the "duality" relation) in order to have a palindromic structure generating the tiling canon (instead of governing the entrances of voices).

Vieru's modal theory represents a remarkable example of an algebraically oriented perspective on intervallic thinking in music theory, analysis, and composition. It is easy to show [5] that the concept of the composition of modal structures is equivalent to the "transpositional combination" of the set-theoretical tradition [13]. These two equivalent constructions not only help the music analyst to decompose musical structures into elementary blocks—such as in the case of Messiaen's "modes with limited transpositions"—but they also provide the general framework for serial techniques such as Pierre Boulez's chord multiplications [24, 37].

Figure 2 shows how the "transpositional combination" operation is connected to the construction of rhythmic tiling canons. This and the following musical examples have been realized using the OpenMusic visual programming language [1]. In this functional programming language, all musical operations are represented in a graphical way and the user simply connects outputs of a given function (or object) with inputs of a second function (or object). The results of the operations can be represented, as in Figure 2, with geometric objects (such as the circular representation) or in standard notation.[f]

Tiling rhythmic canons such as those shown in Figure 2 are easy to obtain, since one may simply make use of some well-known results in group theory, such as the following one, which is the application of the Fundamental Theorem of Finite Abelian Groups to the special case of cyclic groups:

> *Theorem: a cyclic group of order* n *is the direct sum of its maximal subgroups.*

In most cases, the cyclic group of order 12 is simply the equal-tempered system of pitch-classes (hence the notes contained within an octave). The musical interpretation of this decomposition theorem in terms of a transpositional combination produces a tiling of the pitch space with an augmented triad and its four transpositions (including the identity transformation) or, equivalently, with a diminished seventh chord and its possible transpositions (three, by also including, as in the previous case, the identity transformation). From a rhythmic perspective, this decomposition leads to a tiling rhythmic canon in three or four voices, depending on the choice of the factor of the group decomposition as the inner rhythm. Figure 3 shows the two "dual" canons obtained through the rhythmic interpretation of the previous

[f]For more examples of computer-aided models of tiling canbonic structures in OpenMusic, see [2]

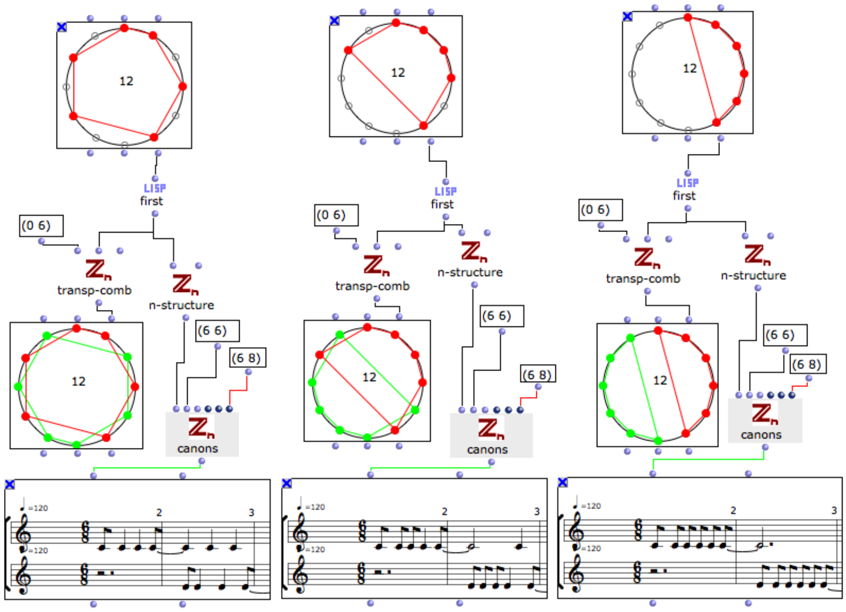

Fig. 2. Generation of three different tiling rhythmic canons via the transpositional combination process, starting from three palindromic rhythmic structures.

decomposition theorem.

Applying the decomposition theorem to the tiling process is an easy way to obtain special classes of tiling rhythmic canons in which *both* the inner and the outer voices correspond to regular patterns. The series of papers published by Dan Vuza in *Perspectives of New Music* from 1991 to 1993 not only constitute a milestone in the development of the mathematical theory of tiling canons but also offer new possibilities for composers to free themselves from this regularity constraint. Among the rich collection of new, interesting music-theoretical models introduced by Vuza, the concept of “Regular Complementary Canons of Maximal Category” describes canons having the remarkable property of tiling the time axis without inner periodicity (Vuza 1991-1993). From an algebraic point of view these canons, currently known as “Vuza Canons”, correspond to a factorization of a cyclic group into two non-periodic *subsets*. These types of factorizations are fascinating objects for mathematicians and they appeared in mathematical treatise well before Vuza’s papers. As an interesting example, one may mention the decomposition proposed by László Fuchs in his monograph on

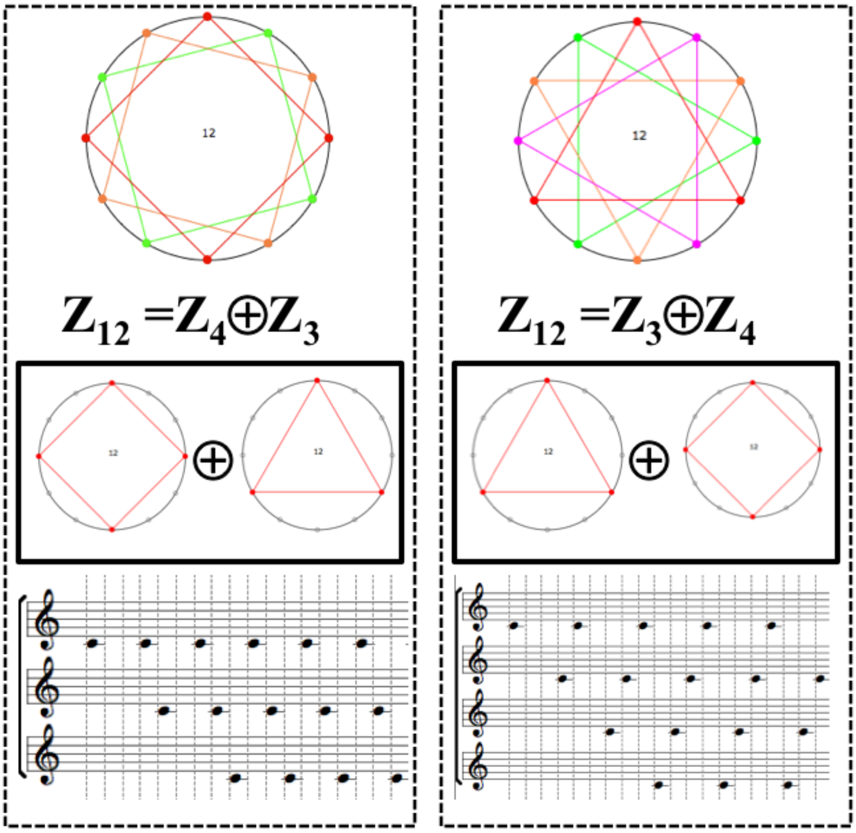

Fig. 3. Two "dual" canons obtained via the decomposition theorem.

abelian groups [16] assuring that the cyclic group with seventy-two elements can be decomposed as a sum of two subsets whose period is equal to the order of the group (i.e. seventy-two). By virtue of this fact, François Le Lionnais—one of the founding members of the French *Oulipo* group (*Ouvroir de littérature potentielle*)—grants a special place to the number 72, which was therefore included in the encyclopedia of remarkable numbers compiled in collaboration with mathematician Jean Brette [22]. This corresponds to the particular Vuza canon of period 72 shown in Figure 4.

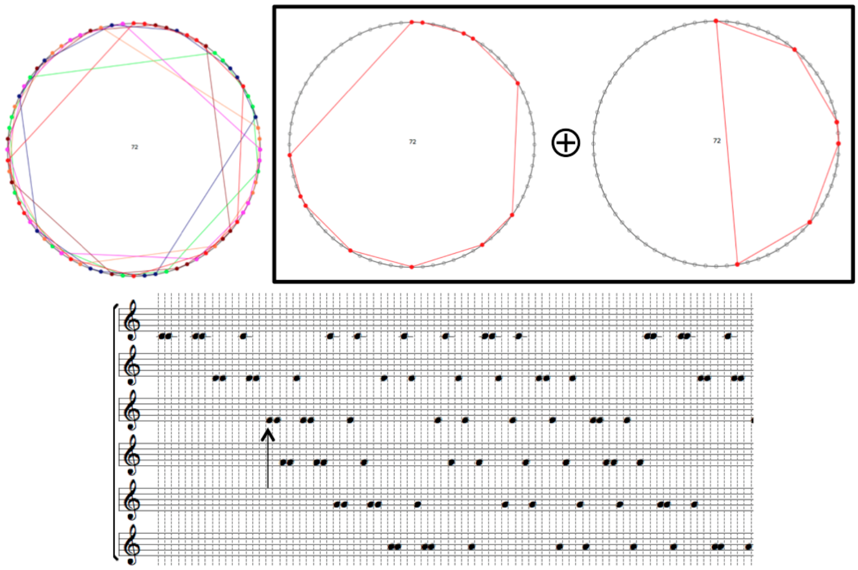

Fig. 4. A Vuza canon obtained by using the decomposition of a cyclic group into the direct sum of two subsets resulting in a period equal to the "remarkable" number 72. Note that the time axis is tiled from the beginning of the third voice, as indicated in the score with an arrow.

2. The computational model

Vuza canons are difficult to obtain and we have at the moment no exhaustive algorithm providing the complete list of these musical structures. Nevertheless, several tools exist which have been integrated in the MathTools environment of OpenMusic visual programming language enabling the composers to produce several classes of tiling rhythmic canons by means of constraint programming, group factorizations, and polynomial representations [2]. Examples of tiling rhythmic canons include canons by translation (from the simplest cases to the cyclotomic canons and Vuza canons) and by augmentation (i.e., canons obtained by affine transformations). This last family of rhythmic tiling canons is very interesting from a compositional and perceptual point of view, since the composer has much more freedom in the selection of the rhythmic pattern and the corresponding "stretching" factors which allow the time axis to be tiled by augmentations (Figure 5). This model was originally proposed by mathematician and music-theorist Thomas Noll [28] and successively integrated in the OpenMusic MathTools environment [7].

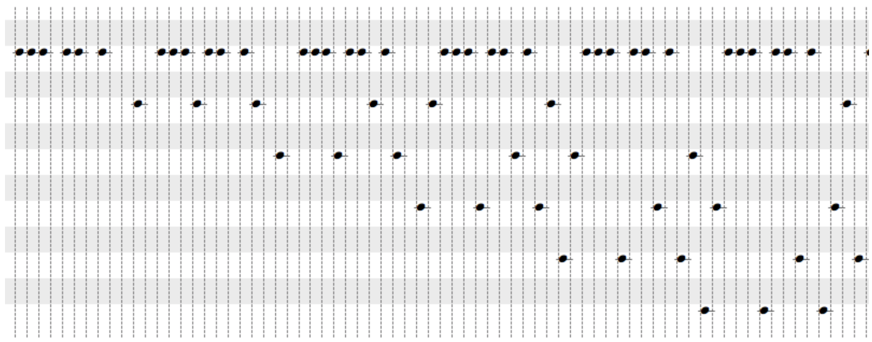

Fig. 5. A tiling canon obtained by augmentation by factor five of the rhythmic pattern R.

Canonic tiling structures which are obtained and represented within the OpenMusic visual programming language are not limited to twelve-tempered equal systems, as the following example shows (Figure 6). In fact, we can map the information of the inner rhythmic pattern into a given microtonal space, such as the twelfth-tone division of the octave which has the same underlying algebraic structure of the inner rhythm, i.e. it is isomorphic to the cyclic group of order 72. This operation is a prototypical example of a "transfer of structure" between the rhythmic and the pitch domains whose cognitive implications still constitute a source of debate in the field of music theory and musicology.[g]

3. Some compositional applications

Surprisingly, in spite of the rigid form of rhythmic tiling canons, the establishment of a catalogue of solutions, which have been made available in OpenMusic, has surely played a major role in generating interest among composers in this theoretical model. Every composer with whom we had the opportunity to collaborate interpreted the catalogue of solutions in a different way, leading to a variety of stylistically very different compositional projects. We briefly present four examples of compositional uses of rhythmic tiling canon constructions showing the diversity of compositional strategies starting from a common theoretical model.

[g]See [23, 30] for two opposite opinions with respect to the problem of the pitch-rhythm correspondence.

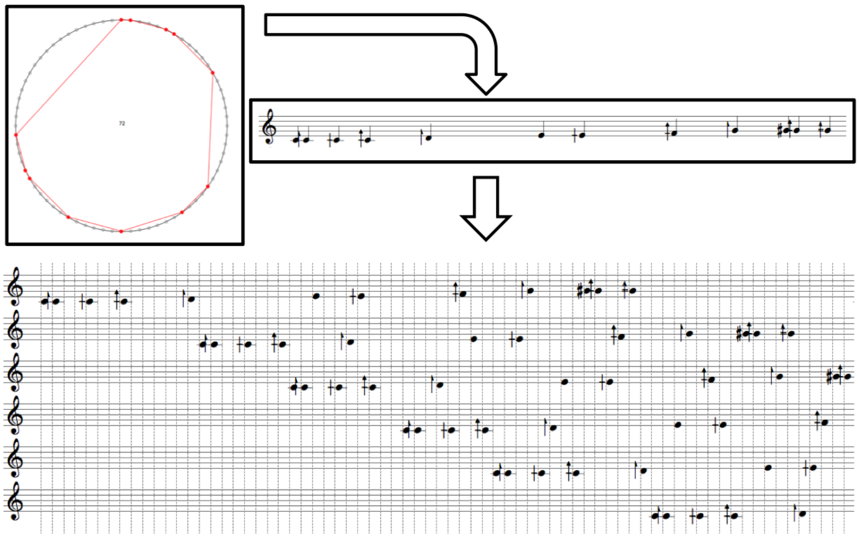

Fig. 6. An example of microtonal tiling canon in OpenMusic

3.1. *Fabien Lévy: morphological approaches and pedagogical strategies in tiling canon constructions*

The first compositional application of Vuza canons was made by composer Fabien Lévy in his orchestral piece *Coïncidences* (1999). Complex musical objects filled the underlying rhythmic grid provided by a Vuza canon in such a way that the global perceptual result is not heard contrapuntally but rather as a continuous information flow where "timbral melodies" [21] spontaneously emerge via the combinatorial play of the different voices of the canon. The composer further used this canonic construction as a pedagogical device, and for the generalization of canon construction beyond temporal translations to augmentations. To the "pedagogical" class of pieces belongs the cycle *Où niche l'hibou* [where the owl nests] (2001), which explores the transposition of a motive into different registers, like in the final piece "Pour la classe" [For the class]. In the computer-aided "meta-work" entitled *Soliloque sur [X, X, X et X]*, as Fabien Lévy defines it [21], the composer builds a "computer's commentary on a concert it misunderstood," according to the subtitle. Recently, Lévy explored the pedagogical implications of Vuza canons much further, in particular with the piece *Als Gregor und Griselda* (2015) for (a not necessarily professional) choir in 6 voices. Figure 7 shows an excerpt of the piece where the tiling canon is presented by stressing its cyclic character instead of its linear one. The six voices of the canon, whose initial attack-point represented by a circle, enter according to the outer durational rhythm (8 8 2 8 8 38).

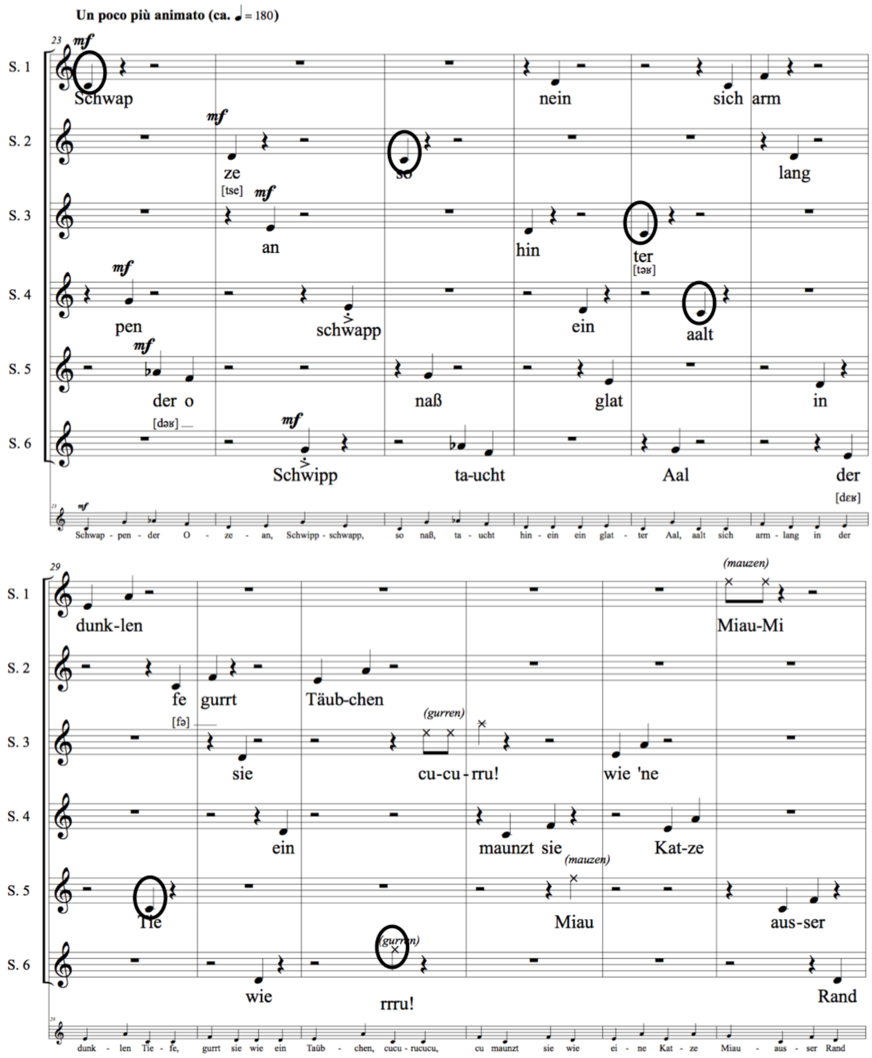

Fig. 7. The Vuza canon utilized by Fabien Lévy in the piece *Als Gregor und Griselda* for a choir, together with an excerpt of the score (reproduced with the kind permission of the composer).

3.2. *Georges Bloch: some practical problems arising from Vuza canons*

The construction of Vuza Canons have found a variety of applications in Georges Bloch's compositional projects, ranging from the piece *Empreinte sonore pour la Fondation Beyeler* (2001), a guided musical tour for an exhibition of the Beyeler Foundation in Basel, Switzerland, to the recent experiments in computer-aided improvisations using the OMax program developed at IRCAM and combining OpenMusic formal models and Max/MSP real-time functions. As rightly observed by the composer in a very detailed analytical account of his compositional practices [9], when using the Vuza canons a composer has to face several questions which arise from aesthetic choices, but which are also linked sometimes to practical and technical problems. One of the most original ideas used by the composer in many of his pieces is the fact of reducing several voices to a single voice obtained by projecting the onsets content of the various voices into one single line.. This can eventually be a necessary strategy that the composer has to adopt when the number of players, as in the case of the project at the Beyeler Foundation in Basel, is less than six, this value being the minimum number voices in a given Vuza canon.[h] Finally, and more anecdotally, the Vuza canon model can enable a composer to "improve" a composition process such as Messiaen's pseudo-tiling construction used in the piece *Harawi* mentioned at the beginning of this chapter. Georges Bloch's piece *Harawun*, a new realization of *Harawi* that more strictly adheres to a tiling canonic form, shows how Vuza canons can be useful in this reconstruction process (Figure 8).

3.3. *Mauro Lanza: exploring the partial redundancy of Vuza canons*

A third example of a composer having benefitted from a computational model of Vuza canons is Mauro Lanza, who was inspired by the local periodicity of some of the factors in the case of cyclic groups with large cardinality. In his piece entitled *La descrizione del diluvio* (2007), for choir and

[h] The two other practical problems linked to Vuza canons are (1) the relationship between canons and (2) the duality continuum versus texture. In the first case, the question is whether it is possible to (rhythmically) "modulate" between canons of different sizes and different numbers of voices. The second question arises from the way in which Vuza canons and, more generally, tiling canons are perceived. Far from the theoretical model which guarantees a continuous line, the use of different instruments for the voices having different characteristics in terms of resonances and attack times begs a more "texture-oriented" aural perception rather than "pointwise" analytical listening.

Fig. 8. The piece *Harawun* by Georges Bloch as an "exact" version of Messiaen's *Harawi* where the underlying inner rhythm has been adapted in order to fit with the model of Vuza canons (used with the kind permission of the composer).

electronics, he uses a particular Vuza canon of period 392 built on an inner rhythm of cardinality 28 and in which the fourteen voices enter according to a non-invertible rhythm. Although in Vuza canons the inner rhythm has no inner periodicity, it is possible to find local repetitions of shorter rhythmic patterns of various lengths. This suggested to the composer that he selects the notes and the durations to emphasize these quasi-periodicities of the Vuza canon, which provides some redundancy within each voice. According to the composer, in this piece, "6 voices are live and 8 are in the electronic part". The choice of the notes and the durations was made in such a way as to stress some quasi-periodicities of the underlying Vuza canon and this gives to each voice a much more "redundant" character" (Figure 9).

3.4. *Daniele Ghisi: quantifying Vuza Canons*

The last example of a compositional process using the structure of Vuza canons is provided by composer Daniele Ghisi in his work *La notte poco prima della foresta* (2009), a chamber opera for an actor, a mezzo-soprano,

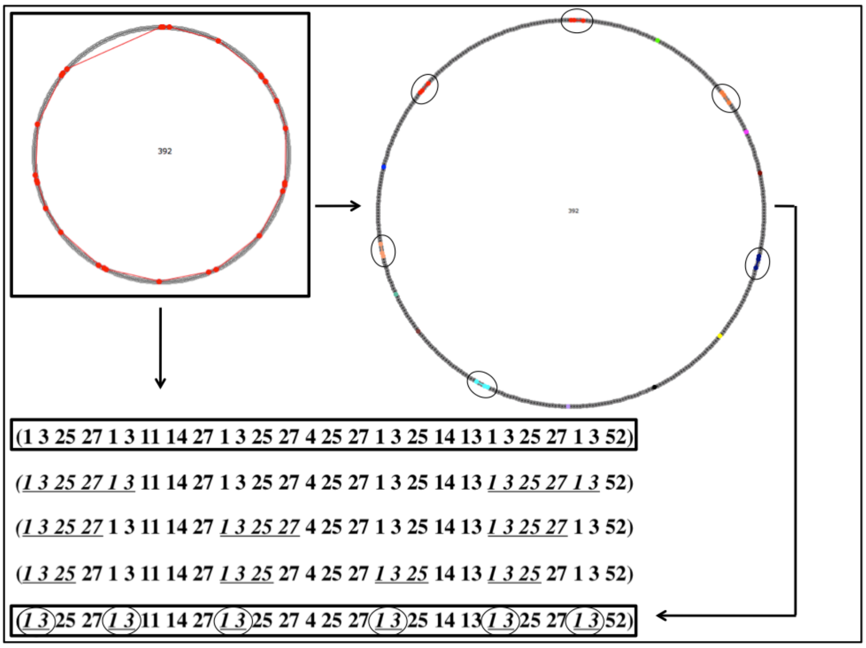

Fig. 9. The Vuza canon of period 392 used by composer Mauro Lanza in his piece *La descrizione del diluvio*. The inner rhythm is represented in a way which stresses the repetition of small rhythmic cells, which gives a kind of redundancy to this aperiodic structure.

a baritone, an instrumental ensemble and electronics. In this piece, a 14-voice Vuza canon of length 168 is utilized, which is processed via a quantification algorithm in order to be transcribed in common music notation. Figure 10 shows a patch in OpenMusic which contains the implementation process, from the two original patterns tiling the space (via a transpositional combination).

3.5. *Tom Johnson: from Vuza to "Perfect Tiling Canons"*

It would be hard to end a survey chapter on tiling rhythmic canons without mentioning Tom Johnson's compositional and theoretical contributions to the field. Although he has never used the structure of Vuza canons in his compositions, Vuza's original theoretical contributions have played a fundamental role in Tom Johnson's compositional activity since the end of the 1990s. As Johnson recognizes in a self-analytical essay on tiling structures in his music, Vuza's article published in *Perspectives of New Music* on "Regu-

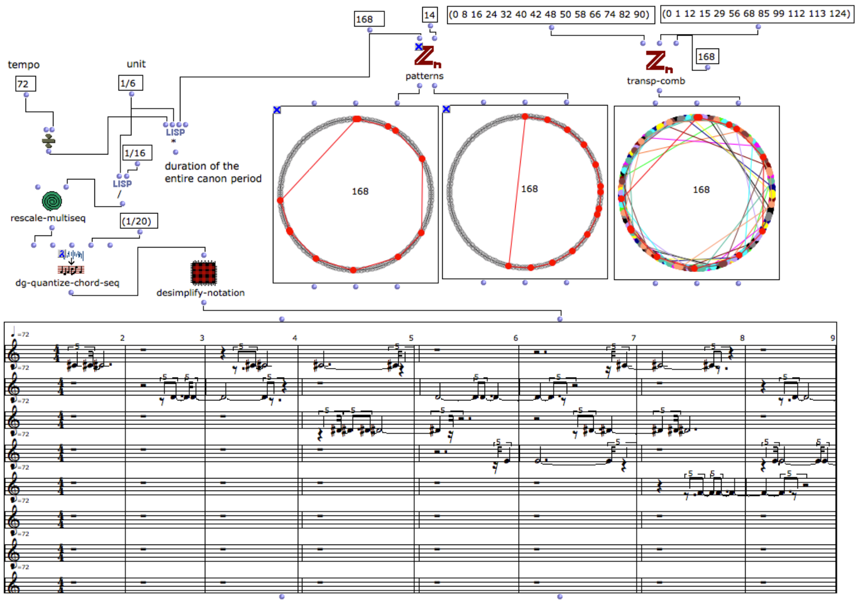

Fig. 10. A OpenMusic patch used by Daniele Ghisi for constructing the Vuza canon used in his piece *La notte poco prima della foresta* (reproduced with kind permission of the composer).

lar Complementary Unending Canons of Maximal Category" is, in his eyes, "the most important music theory treatise of the last 20 years, particularly since it is one of those rare cases where music theory has preceded musical practice" [20, p. 10]. Because of the large period (at least 72) and the number of voices of such canons (minimum six), there is no surprise that this theoretical model could not find a natural place in the universe of a minimalistic composers such as Tom Johnson. Nevertheless, some extensions of Vuza's model, in particular the family of augmented canons, have been widely explored by Johnson, who introduced the class of "Perfect Tilings", i.e. tilings having a different tempo for each voice. The piece *Tilework for Piano* (2003) is the first example of such a family of augmented canons. The formal structure is obtained by five different augmentations of a rhythmic pattern of three elements, hence the subtitle "perfect triplet tilings of 5th order" that one finds in the score (Figure 11).

Despite the apparently simple structure of this canonic construction, there remains some interesting open problems connected with perfect rhythmic tilings, in particular once the construction is formalized in graph-

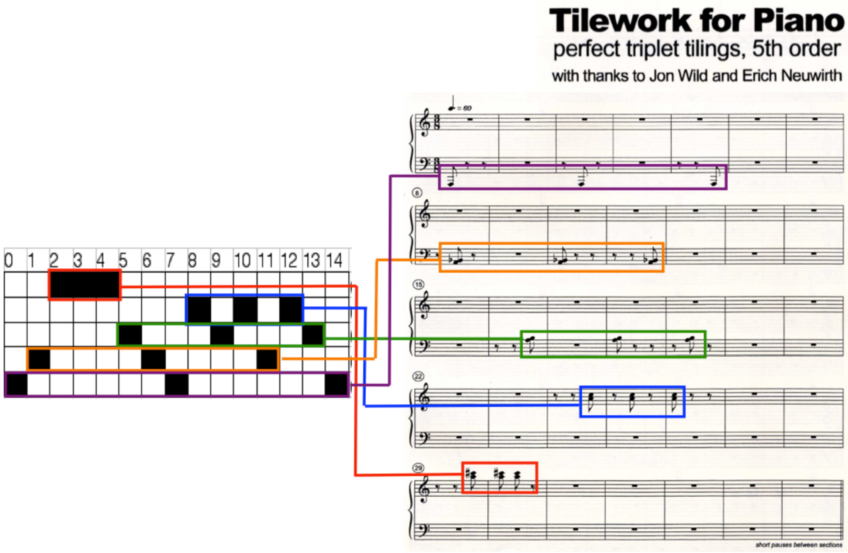

Fig. 11. A perfect canon used by Tom Johnson in his piece *Tilework for piano* (2003), together with the composer's underlying grid (reproduced with kind permission of the composer).

theoretical terms [14].

4. Conclusions

This short description of the history of rhythmic tiling canons and the compositional applications of a very special and constrained class of canons, namely Vuza canons, clearly shows the importance of connecting theoretical research, compositional practice, and computational modeling. We focused on the Western music tradition, but interesting problems arise when one approaches these music-theoretical constructions from an ethnomusicological and ethnomathematical perspective [12, 19]. Moreover, as suggested earlier in our description of "mathemusical problems", a music-theoretic construction may intersect with a number of different mathematical problems. We have discussed elsewhere [6] the surprising connection between rhythmic tiling canon construction and the Minkowski-Hajós problem [18, 26, 27, 33] as well as the second parallel development of the theory of tiling that originated in a problem raised by Bent Fuglede in functional analysis [17]. This conjecture states that there is an equivalence between the *spectrality* of a set and its tiling character. Vuza canons are precisely the musical constructions that could help mathematicians formulate an answer to this open conjecture. In fact, for the one-dimensional case (which is still an open problem, together with the two-dimensional case), all tiling canons which are not Vuza canons have the spectral property [3, 4]. This means that a possible counterexample of the spectral conjecture may already exist within the yet unwritten pages of the catalogues of all possible (and still unheard) Vuza canons. This motivates our efforts to finally obtain a complete enumeration and classification of Vuza canons.[i]

Acknowledgment

The problem of constructing tiling canons has motivated my interest in mathematics and music since the late 1990s and I am happy to take this opportunity to express my gratitude to all the friends and colleagues with whom I have been able to share my interest in this fascinating "mathemusical" subject. In particular, I am deeply grateful to Carlos Agon, for introducing me to the computational universe of the OpenMusic Visual Programming Language and for contributing to the MathTools project, a large part of which is devoted to tiling canon construction. Many thanks to all the mathematicians, music theorists and composers who have enabled this initially exotic topic to grow and become a productive axis in "mathemusical" research. All this would not have been possible without the initial intuition of Dan Tudor Vuza, whose contributions in the field still stimulate an entire community of scholars. Last but not least, I would like to thank Elaine Chew and Jordan Smith for their critical comments on a preliminary version of this article whose balance between the mathematical content and the general audience was far from being adequate. Needless to say, the author is the only responsible for the remaining hermetic constructions and passages in the text.

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[i]For the general problem of enumerating tiling (not necessarily Vuza) canons, see [15]. See [10, 11] for an alternative approach to this classification problem starting from the concept of modulus p canons.

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